THE COMPUTATION OF ULTRAPOWERS BY SUPERCOMPACTNESS MEASURES

John C. Smith III, B.S., M.S.

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APPROVED:

Stephen Jackson, Major Professor
Paul Lewis, Committee Member
Elizabeth Bator, Committee Member
Neal Brand, Chair of the Department of Mathematics
C. Neal Tate, Dean of the Robert B. Toulouse School of
Graduate Studies

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The results from this dissertation are a computation of ultrapowers by supercompactness measures on $P_{\omega_1}(\omega_m)$, $m \in \omega$, and concepts related to such measures.

The second chapter gives an overview of the basic ideas required to carry out the computations. Included are preliminary ideas connected to the measures, W^1_m, S^1_1 and the supercompactness measures, μ_κ, $\kappa \in O.N.$. Order type results are also considered in this chapter with the key theorem being that for a c.u.b. set C, $\forall^*_{\mu_\kappa} S \;\; o.t.(S) \in C$.

In chapter III we give an alternate characterization of μ_2 using the notion of iterated ordinal measures. Basic facts related to this characterization are also considered here.

The remaining chapters are devoted to finding bounds for $j_{\mu_m}(\omega_n)$ with arguments taking place both inside and outside the ultrapowers. Conditions related to the upper bound are given in chapter VI.

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CHAPTER 1

INTRODUCTION

This paper is an effort to bring to light some properties and results associated to the supercompactness measures on certain sets. The specific sets in question are the sets $P_{\omega_1}(\omega_n)$ where $n \in \omega$. By definition, $P_{\omega_1}(\omega_n)$ is the collection of countable sets of ordinals less than ω_n. In other words

$$P_{\omega_1}(\omega_n) = \{S \mid S \subset \omega_n \text{ and } |S| < \omega_1\}.$$

The results in this paper will be obtained using The *Axiom of Determinacy* (AD). This axiom asserts that for any two player integer game, one of the players has a *winning strategy*. To understand this axiom, we must define what is meant by the term *game*. We first consider the set of infinite sequences of integers ω^ω where we view ω^ω as the countable product of ω with the discrete topology. This space, known also as the *Baire Space*, is homeomorphic to the irrational numbers (see [Wi] or [HW]) and is a complete, separable metric space. For the purpose of this paper the elements of ω^ω will be referred to as real numbers. Letting $A \subset \omega^\omega$ we define the game G_A by having the two players, player I and player II, alternately *play* integers to produce an element $x \in \omega^\omega$. We say that player I wins the game if and only if $x \in A$. AD guarantees that in such a game one of the two players has a winning strategy, that is, one of the two players can always win the game regardless of what the other player does. At first glance this axiom seems to be very ill-motivated. It has been shown, however, that granting AD many beautiful structural results may be obtained. For instance, every set is absolutely measurable, every set has the property of Baire, and so on. For a full discussion of this axiom and its consequences, see [Kl] and [M2]. The problem with assuming AD is that in V, the ground model of ZFC, AD is false. In fact it was shown by Gale and Stewart that there exist games that are not determined. The proof requires a blatant appeal to the axiom of choice (see [GS]). One is led to question the validity of such an axiom, but its utility lies in the fact that often the results obtained using AD may be transferred back into V.

Several different *measures* will be defined and used throughout this paper. By measure we will mean a countably additive *ultrafilter*. Given a set X, an ultrafilter ν on X and $B \subset X$, we will say that B has ν-measure 1 if and only if $B \in \nu$. Then by the properties of ultrafilters if $B \in \nu$ and $A \supset B$ then $A \in \nu$. A consequence of AD is that any ultrafilter is countably additive i.e., any ultrafilter is a measure. Using these measures the notion of an *ultrapower* may be defined. Let X and Y be sets with ν a measure on X. Let $\mathcal{A}$ be the set of all functions from X to Y. We may describe an equivalence relation $\sim$ on such functions by

$$F \sim G \iff \{x \in X \mid F(x) = G(x)\} \in \nu.$$

Furthermore, assuming an ordering $\prec$ on the set Y, we may define an ordering $\ll$ on such functions by

$$F \ll G \iff \{x \in X \mid F(x) \prec G(x)\} \in \nu.$$

For more concerning ultrapowers see [Dr] or [Je]. It will always be the case (in this paper) that the set Y is a set of ordinals. For example, in the case where $Y = \omega_1$ the ultrapower $j_\nu(\omega_1)$ is defined to be the classes of all functions which are *less than* the constant function $C(x) = \omega_1$. The reason for studying ultrapowers is that one may describe ordinals by classes of functions. The usefulness of this idea may not seem obvious at first glance but the power of such representations will be realized by the results proven throughout this and other papers dealing with ultrapowers.

In chapter II we will introduce all of the measures to be considered and present the reader with some of the properties associated to the respective measures. In chapter III we will concentrate on μ_2, the supercompactness measure on $P_{\omega_1}(\omega_2)$. Chapters IV and V will provide lower bound results for the ultrapowers, $j_{\mu_m}(\omega_n)$. Chapter VI will concentrate on what is known concerning the upper bound.

CHAPTER 2

PRELIMINARY RESULTS

The purpose of this chapter is to acquaint the reader with the different measures which will be used throughout this paper. The relationships that the measures share with each other and properties related to specific measures will also be considered.

2.1 The Normal Measure on ω_1

A measure which will be frequently used is the normal measure W_1^1 on ω_1. The measure is induced by closed, unbounded subsets (c.u.b. sets) of ω_1. A c.u.b. set C is defined to be a set which is, as indicated by the name, unbounded in ω_1 and closed with respect to increasing ω-sequences of elements of C. The measure may then be described by

$$A \in W_1^1 \iff \exists C, \text{ a c.u.b. set, such that } C \subset A.$$

This measure is, in fact, a normal measure on ω_1 where by normal we mean that any function which presses down on a measure 1 set is constant on a measure 1 set. That is, $\forall H : \omega_1 \to \omega_1$

$$\{\alpha \mid H(\alpha) < \alpha\} \in W_1^1 \Rightarrow \exists \beta \ \ \{\alpha \mid H(\alpha) = \beta\} \in W_1^1.$$

A consequence of this measure is the *Strong Partition Property.* This property states that if P is a function taking increasing functions f of some *specified type* into the set

$\{0,1\}$ then there is a c.u.b. set C so that P is constant on the set

$$\{f \mid f : \omega_1^n \to C \text{ and } f \text{ is of the correct specified type}\}.$$

The following definition will shed some light on the notion of specified types.

Definition 2.1 *Let $f : \omega_1 \to \omega_1$ be an increasing function. Then f is a function of ω-correct type (abbreviated c.t.) if f satisfies the following two conditions:*

1. *f has uniform cofinality ω. That is, there is some function $\hat{f} : (\omega \times \omega_1) \to \omega_1$ which is increasing on the first coordinate such that*

$$\forall \alpha \in \omega_1 \quad f(\alpha) = \sup_{n \in \omega} \hat{f}(n, \alpha).$$

   We say that $\hat{f}$ induces f.

2. *f is everywhere discontinuous. That is $\forall \alpha \quad f(\alpha) > \sup_{\beta < \alpha} f(\beta)$.*

This definition can be extended to include functions from ω_1^n to ω_1. We consider the set

$$\{(\alpha_1, \alpha_2, \dots, \alpha_n) \mid \alpha_1 < \alpha_2 < \dots < \alpha_n\}$$

with the following modified lexicographic ordering:

$$
\begin{array}{rl}
& (\alpha_1, \alpha_2, \ldots, \alpha_n) <_n (\beta_1, \beta_2, \ldots, \beta_n) \\
\iff & (\alpha_n, \alpha_1, \ldots, \alpha_{n-1}) <_{lex} (\beta_n, \beta_1, \ldots, \beta_{n-1}) \\
\iff & \alpha_n < \beta_n \\
\text{or} & \alpha_n = \beta_n \text{ and } \alpha_1 < \beta_1 \\
\text{or} & (\alpha_n, \alpha_1) = (\beta_n, \beta_1) \text{ and } \alpha_2 < \beta_2 \\
\text{or} & \vdots \\
\text{or} & (\alpha_n, \alpha_1, \ldots, \alpha_{n-2}) = (\beta_n, \beta_1, \ldots, \beta_{n-2}) \text{ and } \alpha_{n-1} < \beta_{n-1}
\end{array}
$$

We will use the abbreviation $f :<_n \to \omega_1$ to refer to a function taking increasing sequences of elements of ω_1 having length n, to ω_1. Then one might say that the domain of $<_n$ is $\{(\alpha_1, \alpha_2, \ldots, \alpha_n) \mid \alpha_1 < \alpha_2 < \cdots < \alpha_n\}$. As an extension of Definition 2.1 we have the following

Definition 2.2 *A function $f :<_n \to \omega_1$ is of ω-correct type if it satisfies the following three conditions.*

1. *f is of uniform cofinality ω. That is, there is some $\hat{f} : (\omega \times \omega_1^n) \to \omega_1$ increasing on the first coordinate such that*

$$\forall(\alpha_1, \alpha_2, \ldots, \alpha_n) \quad f(\alpha_1, \alpha_2, \ldots, \alpha_n) = \sup_{n \in \omega} \hat{f}(n, \alpha_1, \alpha_2, \ldots, \alpha_n).$$

2. *f is totally discontinuous. That is $\forall \vec{\beta}$*

$$\sup\{f(\vec{\alpha}) \mid \vec{\alpha} <_n \vec{\beta}\} < f(\vec{\beta}).$$

3. f is increasing. So if $\vec{\alpha} <_n \vec{\beta}$ then $f(\vec{\alpha}) < f(\vec{\beta})$.

Given any function $f :<_n \to \omega_1$ we may define a related function called the *first invariant of f* and denote this function $f(1)$. For $\beta < \omega_1$ we define

$$f(1)(\beta) = \sup_{\alpha_0, \alpha_1, \ldots, \alpha_{n-1} < \beta} f(\alpha_0, \alpha_1, \ldots, \alpha_{n-1}, \beta).$$

Then notice that $f(1) : \omega_1 \to \omega_1$ and

$$\forall (\alpha_0, \alpha_1, \ldots, \alpha_n) \;\; f(\alpha_0, \alpha_1, \ldots, \alpha_n) \leq f(1)(\alpha_n).$$

Definition 2.3 (W_1^n) *Let A be a set of n-tuples of ordinals in ω_1. Then*

$$A \in W_1^n \iff \exists C, \text{ a c.u.b. set, such that } \forall \vec{\alpha} \in C^n \;\; \vec{\alpha} \in A.$$

It is the case that W_1^n is a measure but for $n > 1$ the measure is not normal.

Using the above measures we may define *classes of functions* in the following way. We say that $f \sim g$, where $f, g : \omega_1^n \to \omega_1$, if there is a c.u.b. set C so that $\forall \vec{\alpha} \in C^n \;\; f(\vec{\alpha}) = g(\vec{\alpha})$. Then we may define the class of f, $[f]_{W_1^n} = \{g \mid g \sim f\}$. We will often suppress the measure W_1^n when referring to a class where there is no fear of confusion as to which measure is being used. An important observation concerning functions of uniform cofinality ω is that $[f]$ is an ordinal of cofinality ω if and only if $f :<_n \to \omega_1$ is a function of uniform cofinality ω. This will be demonstrated later (Lemma 2.16).

The functions taking ω_1^n to ω_1 yield some interesting and useful results. The following lemmas demonstrate some of these.

Lemma 2.4 *Suppose that $g_1, g_2 : \omega_1^n \to \omega_1$ are functions such that there is some c.u.b. $C \subset \omega_1$ so that whenever $(\alpha_1, \alpha_2, \dots, \alpha_n) \in C^n$*

$$g_1(\alpha_1, \alpha_2, \dots, \alpha_n) = g_2(\alpha_1, \alpha_2, \dots, \alpha_n).$$

If $f_1, f_2 : \omega_1 \to C$ are increasing functions such that $f_1 \sim f_2$ then there is some $\hat{C}$, a c.u.b. set, so that $\forall(\alpha_1, \alpha_2, \dots, \alpha_n) \in \hat{C}^n \quad \forall 1 \le k \le n$

$$g_1(\alpha_1, \dots, \alpha_{k-1}, f_1(\alpha_k), \alpha_{k+1}, \dots, \alpha_n) = g_2(\alpha_1, \dots, \alpha_{k-1}, f_2(\alpha_k), \alpha_{k+1}, \dots, \alpha_n).$$

Proof. Let g_1, g_2, C, f_1 and f_2 be as in the hypothesis. Let C_1 be the c.u.b. set on which f_1 and f_2 agree. Let $\hat{C} = C_1 \cap C$. Then if $(\alpha_1, \alpha_2, \dots, \alpha_n) \in \hat{C}^n$ both $f_1(\alpha_k)$ and $f_2(\alpha_k)$ are in C. Moreover, $f_1(\alpha_k) = f_2(\alpha_k)$. Then

$$g_1(\alpha_1, \dots, \alpha_{k-1}, f_1(\alpha_k), \alpha_{k+1}, \dots, \alpha_n) = g_2(\alpha_1, \dots, \alpha_{k-1}, f_2(\alpha_k), \alpha_{k+1}, \dots, \alpha_n).$$

∎

Lemma 2.5 *Let $f : \omega_1 \to \omega_1$. Then there is some c.u.b. set C which is* closed *under f. That is*

$$\forall \alpha \in C \quad (\beta < \alpha \Rightarrow f(\beta) < \alpha).$$

Proof. Fix $f : \omega_1 \to \omega_1$. If $range(f)$ is bounded then easily the result is true so we may assume that $range(f)$ is unbounded in ω_1. For each $\alpha \in \omega_1$ we define $\alpha^0 = \alpha$ and for all $i \ge 1$

$$\alpha^i = \sup_{\beta < \alpha^{i-1}} f(\beta).$$

Then define $\hat{\alpha} = \sup_{i \in \omega} \alpha^i$. Let $\hat{C} = \{\hat{\alpha} \mid \alpha \in \omega_1\}$ and let C be the set of all points which are closed under f. Then $\hat{C} \subset C$ so $C \neq \phi$. Moreover, since $range(f)$ is unbounded so is C. To see that C is closed, consider the sequence $\{\alpha_i\}_{i\in\omega}$ of elements of C with $\alpha = \sup_{i\in\omega}\alpha_i$. Then if $\beta < \alpha$ there is some $i \in \omega$ such that $\beta < \alpha_i$ which implies that $f(\beta) < \alpha_i$ which implies that $f(\beta) < \alpha$ and so $\alpha \in C$.

∎

Lemma 2.6 *Let C be a c.u.b. set. For $\alpha \in \omega_1$ define $\overline{\alpha}$ to be the αth element of C. Then*

$$\exists \hat{C} \subset C \quad \forall \alpha \in \hat{C} \quad \overline{\alpha} = \alpha$$

and $\hat{C}$ is a c.u.b. set.

Proof. It is easily the case that $\forall\alpha \;\; \alpha \le \overline{\alpha}$. Let C_1 be a c.u.b. set closed under $\alpha \mapsto \overline{\alpha}$. Let $\hat{C}$ be the set of limit points for $C \cap C_1$. Fix $\alpha \in \hat{C}$. Then α is a limit ordinal so $\alpha = \sup_{\beta<\alpha}\beta$ and since $\hat{C}$ is closed under $\alpha \mapsto \overline{\alpha}$, $\forall\beta < \alpha \;\; \overline{\beta} < \alpha$. So $\alpha = \sup_{\beta<\alpha}\overline{\beta} \in C$. Thus $\overline{\alpha} = \alpha$. ∎

2.2 The Sliding and Weaving Arguments

Lemma 2.7 (The Sliding Lemma) *Suppose that $f :<_n\rightarrow \omega_1$ and that $\exists C$, a c.u.b. set, so that $f \restriction C^n$ is order preserving. Then $\exists \hat{f} :<_n\rightarrow \omega_1$ so that $[f] = [\hat{f}]$ $\hat{f}$ is order preserving and range$(\hat{f}) \subset$ range(f). Moreover, if $f \restriction C^n$ has uniform cofinality ω then $\hat{f}$ will as well.*

Proof. Fix C such that $f \restriction C^n$ is order preserving. Define $\hat{f}$ as follows:

$$\hat{f}(\alpha_0, \alpha_1, \ldots, \alpha_{n-1}) = f(\overline{\alpha}_0, \overline{\alpha}_1, \ldots, \overline{\alpha}_{n-1})$$

where $\overline{\xi}$ is the ξth element of C. By Lemma 2.6 there is some c.u.b. set $\hat{C} \subset C$ such that

$$\forall \xi \in \hat{C} \;\; \overline{\xi} = \xi.$$

Notice then that

1. $[f] = [\hat{f}]$ since $\hat{C}$ witnesses this.

2. $\hat{f}$ is order preserving. To see this we first observe that if $\alpha < \beta$ then $\overline{\alpha} < \overline{\beta}$. Then by the definition of $<_n$, for $(\alpha_0, \ldots, \alpha_{n-1}), (\beta_0, \ldots, \beta_{n-1}) \in dom(<_n)$

$$(\alpha_0, \ldots, \alpha_{n-1}) <_n (\beta_0, \ldots, \beta_{n-1}) \Rightarrow (\overline{\alpha}_0, \ldots, \overline{\alpha}_{n-1}) <_n (\overline{\beta}_0, \ldots, \overline{\beta}_{n-1})$$

   and since $(\overline{\alpha}_0, \ldots, \overline{\alpha}_{n-1}), (\overline{\beta}_0, \ldots, \overline{\beta}_{n-1}) \in C^n$

$$\hat{f}((\overline{\alpha}_0, \ldots, \overline{\alpha}_{n-1})) < \hat{f}((\overline{\beta}_0, \ldots, \overline{\beta}_{n-1})).$$

3. f is of uniform cofinality ω implies $\hat{f}$ has uniform cofinality ω since

$$\forall (\alpha_0, \ldots, \alpha_{n-1}) \;\; \hat{f}(\alpha_0, \ldots, \alpha_{n-1}) = f(\overline{\alpha}_0, \ldots, \overline{\alpha}_{n-1}).$$

∎

This argument is referred to as the sliding argument since for a given function we are sliding up to a function in the same class which is order preserving everywhere and not just on a c.u.b. set. Closely related to this idea is that of weaving functions.

Lemma 2.8 (The Weaving Lemma) *Suppose that $f, g : \omega_1 \to \omega_1$ are functions such that $\omega_1 < [f] < [g]$. Then there exist functions $\hat{f}$ and $\hat{g}$ so that*

1. $[f] = [\hat{f}]$ *and* $[g] = [\hat{g}]$

2. $\forall \alpha, \beta \in \omega_1 \;\; \alpha < \beta \Rightarrow \hat{f}(\alpha) < \hat{g}(\alpha) < \hat{f}(\beta)$

3. If f *has uniform cofinality* ω *then so does* $\hat{f}$ *and similarly for* g.

Proof. Let C be a c.u.b. set such that $\forall \alpha \in C \;\; \alpha < f(\alpha) < g(\alpha)$, $f \restriction C$ and $g \restriction C$ are order preserving and C is closed under g. Define

$$\hat{f}(\beta) = f(\overline{\beta}) \quad \text{and} \quad \hat{g}(\beta) = g(\overline{\beta})$$

where $\overline{\beta}$ is the (β)th element of C. By Lemma 2.6 there is a c.u.b. set $\hat{C}$ so that

$$\forall \beta \in \hat{C} \;\; \overline{\beta} = \beta$$

thus $[f] = [\hat{f}]$ and $[g] = [\hat{g}]$. Fix $\alpha, \beta \in \omega_1$ such that $\alpha < \beta$. Then $\overline{\alpha} < \overline{\beta}$. Also $\overline{\alpha}, \overline{\beta} \in C$ implies that $\hat{f}(\alpha) < \hat{g}(\alpha)$. Since $\overline{\beta} \in C$ and C is closed under g we have that

$$\hat{g}(\alpha) = g(\overline{\alpha}) < \overline{\beta} < f(\overline{\beta}) = \hat{f}(\beta).$$

As in Lemma 2.7 $\hat{f}$ and $\hat{g}$ are order preserving and if f has uniform cofinality ω then $\hat{f}$ has uniform cofinality ω and similarly for g. ∎

The weaving argument may be extended to include non-unary functions in an obvious manner. A bit of care must be exercised, however, since *weaving* might mean weaving on a single coordinate or having the entire block weave. More will be said concerning this as it is needed. Both the sliding and weaving arguments are due to Jackson [J1].

One reason for the care taken in presenting facts concerning functions on ω_1 is that these functions may be used to represent ordinals less than ω_ω. Indeed if $\theta < \omega_n$ then there is a function $f : \omega_1^{n-1} \to \omega_1$ representing θ. In other words $[f] = \theta$. The proof of this fact is well known. It will, nevertheless, be shown since the idea behind it will be used throughout this paper. Before presenting the proof, however, we must exploit a powerful technique detailed in the next section.

2.3 Scales, Trees and a Theorem of Kunen

Definition 2.9 *A tree T on a set $Y \neq \phi$ is a collection of finite sequences of elements of Y so that if $v \in T$ and u is an initial segment of v $(u \subset v)$ then $u \in T$.*

We say that a function $g : \omega \to Y$ is a *path* through T if

$$\forall n \in \omega \quad (g(0), g(1), \ldots, g(n-1)) \in T.$$

If g is such a path then we say g is in the *body of T* and denote this, $g \in [T]$. If $[T] \neq \phi$ then we say that T is *ill-founded* otherwise T is considered *well- founded.* If T is a tree on $\omega \times Y$ then we define

$$T(x) = \{(v(0), \ldots, v(n-1)) \in Y^{<\omega} \mid (x(0), v(0), \ldots, x(n-1), v(n-1)) \in T\}.$$

Notice that $T(x)$ is a tree on Y. It was shown by Shoenfield[M2] that if G is a $\mathbf{\Pi}_1^1$ subset of the real numbers then there is some tree T on $\omega \times \omega_1$ so that

$$G = p[T] = \{x \mid \exists g \ \ \forall n \ \ (x(0), g(0), x(1), g(1), \ldots x(n-1), g(n-1)) \in T\}.$$

A closely related idea is that of scales.

Definition 2.10 *A scale $\{\varphi_n\}_{n\in\omega}$ is a sequence of norms on G, a set of Reals so that if $\{x_i\}_{i\in\omega}$ is a sequence of elements from G such that*

1. $\exists x \ \ x_i \to x$

2. $\forall n \ \ \exists \lambda_n \ \ \exists j \ \ \forall i > j \ \ \varphi_n(x_i) = \lambda_n$ *that is φ_n is eventually constant*

then $x \in G$ and each norm exhibits the lower semi-continuity property, *that is* $\forall n \ \ \varphi_n(x) \leq \lambda_n$.

A simple and useful fact is that given any tree T on $\omega \times \kappa$ one may easily derive a scale into κ. Likewise, any scale taking a set G into κ gives rise to a tree on $\omega \times \kappa$. Indeed, let T be such a tree. Then for $n \in \omega$ and x a Real define $\varphi_n(x)$ to be the nth component of g where $(x, g) \in [T]$ and g is the left-most branch. This, of course, provided that there is some g so that $(x, g) \in [T]$. Otherwise we say that $\varphi_n(x)$ is undefined.

Similarly if $\{\varphi_n\}_{n\in\omega}$ is a scale taking G to κ we define a related tree by

$$\begin{aligned} & (y(0), g(0), y(1), g(1), \ldots, y(m-1), g(m-1)) \in T \\ \iff \quad & \exists x \text{ extending } y \ \ \forall 0 \leq n \leq m-1 \ \ g(n) = \varphi_n(x). \end{aligned}$$

The following lemma will be required to prove Kunen's Theorem.

Lemma 2.11 *There is a tree U on $\omega \times \omega$ so that*

$$\sup_{x\in\mathbb{R}}\{|U(x)| : U(x) \textit{ is well-founded } \} = \omega_1$$

Proof. Let G be a $\boldsymbol{\Sigma}^1_1$-complete set of reals. Let $U \subset \omega \times \omega$ be a tree projecting to G. Then for any $x \in \omega^\omega$

$$x \in G^c \iff U(x) \text{ is well-founded.}$$

Thus the norm defined by $\varphi(x) = |U(x)|$ is a $\mathbf{\Pi}^1_1$ norm on a $\mathbf{\Pi}^1_1$-complete set of reals. So

$$\sup_{x\in\mathbb{R}}\{|U(x)| : U(x) \text{ is well-founded } \} = \omega_1.$$

∎

A slight improvement on Kunen's original theorem (see [Ke]) will now be presented.

Theorem 2.12 (Kunen) *Let $f : \omega_1 \to \omega_1$. Then there exists $\prec$, a well ordering of ω_1 so that*

$$\forall\alpha \geq \omega \quad f(\alpha) < |{\prec}\restriction \alpha|\,.$$

Proof. Recall that the set WF is a $\mathbf{\Pi}^1_1$ set of reals which code well-founded orderings of the integers ω. So for any $x \in WF$ we have that $|x|$ is the length of the well-ordering coded by x and notice that $0 \leq |x| < \omega_1$. The way in which a particular x codes this ordering is accomplished as follows. First, for any $n \in \omega$, if $n = 2^i 3^j$ we say that $n \in Code$ and $n = \langle i, j\rangle$. If $n \neq 2^i 3^j$ then likewise, $n \notin Code$.

Secondly, if $i, j \in \omega$ and $\ll$ is the relation coded by x then

$$i \ll j \iff x(\langle i, j\rangle) = 1.$$

If $n \notin Code$ then $x(n)$ is arbitrary. Then $x \in WF \iff x$ codes a relation and the relation coded by x is well-founded.

Let T be a tree on $\omega \times \omega_1$ defined by

$$(x(0), \alpha(0), x(1), \alpha(1), \ldots, x(n-1)\alpha(n-1)) \in T$$

$$\iff \qquad \forall i, j \in \omega \text{ such that } \langle i, j \rangle < n$$

$$x(\langle i, j \rangle) = 1 \iff \alpha(i) < \alpha(j).$$

Notice that for any $\xi < \omega_1$ there is some $x \in WF$ with $|x| = \xi$ and some $\vec{\alpha} \in \omega_1^\omega$ so that $(x, \vec{\alpha}) \in [T]$ and $\forall i \;\; \alpha(i) < \xi$. This is easily true since for a fixed ξ there is a well-founded relation x of that length and for any $n \in \omega \quad |n|_x < \xi$.

Since any strategy for player II may be coded by a real number τ (see [M2]) we may define a tree S on $\omega \times \omega \times \omega$ by

$$\forall n \in \omega \;\; (\tau \restriction n, x \restriction n, y \restriction n) \in S \iff \tau[x] = y.$$

Here $\tau[x]$ is meant to represent the real which results when x is played by player I and player II follows the strategy τ. We also define a tree R on $\omega \times \omega \times \omega \times \omega_1 \times \omega$ by

$$(t, a, b, u, v) \in R \iff (t, a, b) \in S \text{ and } (b, v) \in U \text{ and } (a, u) \in T$$

where U is defined as in the previous lemma. For any τ we define $R(\tau)$ to be $\{(a, b, u, v) \mid \forall n \;\; (\tau \restriction n, a, b, u, v) \in R\}$.

Fix $f : \omega_1 \to \omega_1$. Play the following integer game G_f

$$\begin{array}{c|ccccc} \text{I} & n_1 & n_2 & n_3 & \cdots & x \\ \text{II} & m_1 & m_2 & m_3 & \cdots & y \end{array}$$

Player II wins if and only if

$$x \in WF \Rightarrow U(y) \text{ is well-founded and } |U(y)| > f(|x|).$$

Suppose that player I had a winning strategy σ in this game. We first observe that $x \in WF$ or player II wins trivially. Also, σ may be viewed as a continuous function from ω^ω into ω^ω. So $\sigma''(\omega^\omega)$ is a $\mathbf{\Sigma}^1_1$ set, say H. Thus there is some $\gamma < \omega_1$ so that

$$\sup\{f(|x|) \mid x \in H\} < \gamma.$$

Then player II may simply play y so that $|U(y)| > \gamma$ and he wins. Thus player II has a winning strategy, τ_0, in this game.

Now for any $x \in \omega^\omega$,

$$\begin{aligned} & T(x) \text{ is ill-founded} \\ \Rightarrow \quad & U(\tau_0[x]) \text{ is well-founded and } |U(\tau_0[x])| > f(|x|). \end{aligned}$$

The claim is that $R(\tau_0)$ is well-founded. Suppose, towards a contradiction, that $R(\tau_0)$ is ill-founded. Let $(x, y, \vec{\alpha}, z) \in [R(\tau_0)]$. Then $(x, \vec{\alpha}) \in [T]$. In particular, T is ill-founded which implies that $U(\tau_0[x])$ is well-founded. This, in turn, implies that there is no path through $R(\tau_0)$ which is a contradiction.

Fix $\xi < \omega_1$ with $\xi \geq \omega$. Then by our previous observations,

$$\exists x \in WF \ \ \exists \vec{\alpha} \in \omega_1^\omega \ \ \left[|x| = \xi \text{ and } (x, \vec{\alpha}) \in [T] \text{ and } \forall i \ \ \alpha(i) < \xi \right].$$

Let $y = \tau_0[x]$, $v \in U(y)$ and n be an integer such that the length of v is $n + 1$. Then

$$(x \restriction n, y \restriction n, \vec{\alpha} \restriction n, v) \in R(\tau_0) \restriction \xi$$

where

$$R(\tau_0) \restriction \xi = \{(a, b, u, v) \in R(\tau_0) \mid u \in \xi^{<\omega}\}.$$

Notice that ξ must be greater than ω in order to guarantee that this last statement is true otherwise $x \upharpoonright n$, $y \upharpoonright n$ or v might contain an ordinal $n < \omega$ such that $n > \xi$. By our definition of $R(\tau_0) \upharpoonright \xi$ and as a consequence of the payoff for the game

$$f(\xi) < |U(\tau[x])| \leq |R(\tau_0) \upharpoonright \xi|.$$

Let W be the Kleene-Brower ordering on R viewed as a well-ordering of ω_1 (after identifying $\omega^{<\omega} \times \omega^{<\omega} \times \omega^{<\omega} \times \omega_1^{<\omega} \times \omega^{<\omega}$ with ω_1). For $\tau \in \omega^\omega$ we then define $W(\tau)$ to be the ordering on ω_1 corresponding to $R(\tau)$. Let C be closed under the above identification map, that is

$$\forall \xi \in C \quad (t, a, b, u, v) \in R \upharpoonright \xi \Rightarrow \eta < \xi$$

where η is the element of ω_1 identified with (t, a, b, u, v). Then

$$\forall \alpha \in C \quad f(\alpha) < |W(\tau_0) \upharpoonright \alpha|.$$

Define $\hat{f} : \omega_1 \to \omega_1$ by

$$\hat{f}(\alpha) = \sup_{\beta < N_C(\alpha)} f(\beta)$$

where $N_C(\delta)$ is the next element of C after δ. Let τ_1 be a winning strategy for player II in the game $G_{\hat{f}}$. Then

$$\forall \alpha \in C \quad \hat{f}(\alpha) < |W(\tau_1) \upharpoonright \alpha|.$$

For $\alpha \in \omega_1$ define $\tilde{\alpha}$ to be the supremum of all of the elements of C less than α. For any $\alpha \geq \inf C$

$$f(\alpha) \leq \hat{f}(\tilde{\alpha}) < |W(\tau_1) \upharpoonright \tilde{\alpha}| \leq |W(\tau_1) \upharpoonright \alpha|.$$

Let $\prec^*$ be a well-ordering of ω so that $| \prec^* | > \inf C$. Define the ordering $\prec$ on ω_1 as follows: For $\beta \neq 2n$ we define

$$\hat{\beta} = \begin{cases} m & \text{if } \beta = 2m+1 \\ \beta & \text{otherwise} \end{cases}$$

and for all such α, β,

$$\alpha \prec \beta \iff \hat{\alpha} < \hat{\beta} \text{ with respect to } W(\tau_1).$$

For $\alpha = 2m$ and $\beta = 2n$ we say that

$$\alpha \prec \beta \iff m \prec^* n$$

and if $\alpha = 2m$ and $\beta \neq 2n$ we say $\alpha \prec \beta$. Easily $| \prec | \geq |W(\tau_1)|$. So

$$\forall \alpha \geq \omega \quad f(\alpha) < | \prec \restriction \alpha|.$$

∎

Notice if it is the case that

$$\forall \alpha > \omega \quad f(\alpha) < |\prec \restriction \alpha| ,$$

then for each such α one may find a $\beta_\alpha < \alpha$ so that

$$\forall \alpha > \omega \quad f(\alpha) = |\beta_\alpha|_{\prec \restriction \alpha} .$$

Here $|\beta_\alpha|_{\prec\restriction\alpha}$ means the length of the element β_α with respect to the ordering $\prec\restriction \alpha$. Using the normality of W_1^1 there is some β and some c.u.b. set $\hat{C}$ so that

$$\forall \alpha \in \hat{C} \quad f(\alpha) = |\beta|_{\prec \restriction \alpha} .$$

Another well-ordering $\prec^*$ on ω_1 may then be defined by $| \prec^* \restriction \alpha| = |\beta|_{\prec \restriction \alpha}$. It will often be the case that this new well-ordering is the one being utilized and the "Kunen Argument" will often consist of the statement

$$\forall \alpha \in \hat{C} \quad f(\alpha) = | \prec^* \restriction \alpha|.$$

Definition 2.13 *Let $i, n \in \omega$ with $1 \leq i \leq n$. The function $f_i^n : \omega_1^n \to \omega_1$ is defined to be the projection function for the ith component of $\vec{\alpha} = (\alpha_1, \alpha_2, \ldots, \alpha_n)$. In other words $f_i^n(\vec{\alpha}) = \alpha_i$.*

As a first application of Kunen's Theorem we present the following theorem.

Theorem 2.14

$$\forall i, n \in \omega \text{ with } 1 \leq i \leq n \quad [f_i^n]_{W_1^n} = \omega_i$$

Proof. Fix $i, n \in \omega$ as in the hypothesis. For ease of notation we will allow $[f]$ to mean $[f]_{W_1^n}$. Fix $g : \omega_1^n \to \omega_1$ so that $[g] < [f_i^n]$. Then there is some c.u.b. set C_g so that

$$\forall \vec{\alpha} \in C_g^n \quad g(\vec{\alpha}) < f_i^n(\vec{\alpha}).$$

Define a partition P on $n+1$-tuples of ordinals in ω_1 by

$$P(\alpha_1, \alpha_2, \ldots, \alpha_{i-1}, \lambda, \alpha_i, \ldots, \alpha_n) = 1 \iff g(\alpha_1, \ldots, \alpha_n) < \lambda.$$

Let C be homogeneous for P and define $\hat{C}'$ to be the limit points for $C \cap C_g$. Fix an increasing n-tuple $\vec{\alpha}$ of elements of $\hat{C}'$. Then

$$\xi = g(\vec{\alpha}) < f_i^n(\vec{\alpha}) = \alpha_i.$$

Since α_i is a limit point for $C \cap C_g$ there is some $\lambda \in C \cap C_g$ such that $\xi < \lambda < \alpha_i$ (we may assume that $\lambda > \alpha_{i-1}$ as well). Then the $n+1$-tuple $(\alpha_1, \ldots, \alpha_{i-1}, \lambda, \alpha_i, \ldots, \alpha_n)$ witnesses that C is homogeneous for the 1-side.

Let C' be the set of limit points of C. Then

$$\forall \vec{\alpha} \in C'^n \quad g(\vec{\alpha}) < N_C(\alpha_{i-1}).$$

By Kunen's Theorem there is a well-ordering $\prec$ of ω_1 such that

$$\forall \vec{\alpha} \in C'^n \quad g(\vec{\alpha}) < | \prec \restriction \alpha_{i-1}|.$$

At this point we would like to have that $[g] < [f^n_{i-1}]^+$, so we must show that

$$[(\vec{\alpha}) \to | \prec \restriction \alpha_{i-1}|] < [f^n_{i-1}]^+.$$

To do this notice that the well-ordering $\prec$ induces a well-ordering $\ll$ on $[f^n_{i-1}]$ as follows:

$$[g_1] \ll [g_2] \iff \exists C_\prec \ \ \forall \vec{\alpha} \in C^n_\prec \ \ g_1(\vec{\alpha}) \prec g_2(\vec{\alpha}).$$

Then $| \ll | < [f^n_{i-1}]^+$. So $[g] < [f^n_{i-1}]^+$. Since g was chosen arbitrarily, we have that

$$[f^n_i] \leq [f^n_{i-1}]^+.$$

Proceeding inductively we get that $[f^n_i] \leq [f^n_1]^{+(i-1)}$. It must be argued, at this point, that $[f^n_1] = \omega_1$ and we will have the upper bound. Easily $[f^n_1] \leq \omega_1$ since if $g : \omega_1 \to \omega_1$ is such that $[g] < [f^n_1]$ then there is some c.u.b. set C so that $\forall \alpha \ \ g(\alpha) < \alpha$ and thus g is constant on some c.u.b. set. Notice also that for any ordinal $\xi < \omega_1$ there is a function $g_\xi : \omega^n_1 \to \omega_1$ defined by $g_\xi(\vec{\alpha}) = \xi$. The mapping taking ordinals to functions of this form is clearly well-defined and order preserving. Also if g_ξ is such a function then $[g_\xi] < [f_1^n]$ and so $[f_1^n] = \omega_1$.

By a theorem of Martin [J1] if κ has the strong partition property and ν is any measure on κ then $j_\nu(\kappa)$ is a cardinal. In particular, Martin's Theorem tells us that for any $i \leq n$ $j_{W_1^i}(\omega_1)$ is a cardinal. Let $g : \omega_1^{i-1} \to \omega_1$. Any such g induces a function $\hat{g} : \omega_1^n \to \omega_1$ defined by

$$\hat{g}(\alpha_1, \ldots, \alpha_n) = g(\alpha_1, \ldots, \alpha_{i-1}).$$

Let C be a c.u.b. set closed under $\hat{g}$. Then

$$\forall \vec{\alpha} \in C^n \quad \hat{g}(\vec{\alpha}) < f_i^n(\vec{\alpha})$$

thus $[f_i^n]_{W_1^n} > [\hat{g}]_{W_1^n}$ and consequently $[f_i^n]_{W_1^n} > [g]_{W_1^{i-1}}$.

So $[f_i^n]_{W_1^n} \geq j_{W_1^{i-1}}(\omega_1)$. It is easily the case that $j_{W_1^{i-1}}(\omega_1) > [f_{i-1}^{i-1}]_{W_1^{i-1}}$. We must also show that $[f_{i-1}^{i-1}]_{W_1^{i-1}} \geq [f_{i-1}^n]_{W_1^n}$. Let $g : \omega_1^n \to \omega_1$ be such that $[g]_{W_1^n} < [f_{i-1}^n]_{W_1^n}$. By previous partition arguments we know that g only depends on the ordinals $(\alpha_1, \alpha_2, \ldots, \alpha_{i-1})$ and so the mapping $\pi : [f_{i-1}^{i-1}]_{W_1^{i-1}} \to [f_{i-1}^n]_{W_1^n}$ defined by $\pi(g) = \hat{g}$ where

$$\hat{g}(\alpha_1, \alpha_2, \ldots, \alpha_{i-1}) = g(\alpha_1, \alpha_2, \ldots, \alpha_{i-1}, \ldots, \alpha_n)$$

is an embedding. Since

$$[f_{i-1}^n]_{W_1^n} \leq [f_{i-1}^{i-1}]_{W_1^{i-1}} < j_{W_1^{i-1}}(\omega_1) \leq [f_i^n]_{W_1^n} \leq [f_{i-1}^n]^+_{W_1^n}$$

and $j_{W_1^{i-1}}(\omega_1)$ is a cardinal, $[f_i^n]$ is a cardinal and we are done. ∎

It is easy to see that for the arguments above there was no reliance on the n used and as long as $n \geq i$ $[f_i^n] = \omega_i$. In particular, one may observe that if $1 \leq i \leq n \leq m$ then $[f_i^n] = [f_i^m] = \omega_i$.

As a corollary to Theorem 2.14 we have

Corollary 2.15 *For $n \geq 2$, if $\theta < \omega_n$ then there is some $f : \omega_1^{n-1} \to \omega_1$ so that $[f]_{W_1^{n-1}} = \theta$.*

Proof. Fix $\theta < \omega_n$ and notice that $\theta < [f_n^n] = j_{W_1^{n-1}}$ by the preceding theorem. Thus there is some $f : \omega_1^{n-1} \to \omega_1$ so that $[f]_{W_1^{n-1}} = \theta$. ∎

Lemma 2.16 *An ordinal α may be represented by a function $f :<_n \to \omega_1$ of uniform cofinality ω if and only if α is an ordinal of cofinality ω.*

Proof.

$(\Rightarrow)$

Fix α as in the hypothesis and let f represent α. Since $f :<_n \to \omega_1$ has uniform cofinality ω there is some $\tilde{f} : \omega \times \omega_1^n \to \omega_1$ inducing f. The sequence of classes $\{[f_i]\}_{i\in\omega}$ defined by $f_i(\vec{\alpha}) = \tilde{f}(i, \vec{\alpha})$ for each $i \in \omega$ yields a countable sequence of ordinals which converge to $[f]$. Thus α has cofinality ω.

$(\Leftarrow)$

Let $[f] = \alpha$ be an ordinal of cofinality ω. Then there is an increasing sequence $\{[f_i]\}_{i\in\omega}$ of ordinals which converge to $[f]$. Let C be a c.u.b. set such that

1. $\forall \vec{\alpha} \in C^n \quad \forall i, j \in \omega \quad i < j \Rightarrow f_i(\vec{\alpha}) < f_j(\vec{\alpha})$
2. $\forall \vec{\alpha} \in C^n \quad \sup_{i\in\omega} f_i(\vec{\alpha}) = f(\vec{\alpha})$

Such a C exists since for (1) the sequence of functions is strictly increasing. Also, since $[f] = \sup_{i\in\omega}[f_i]$ there is a C to witness condition (2).

Then $f \upharpoonright C$ is easily of uniform cofinality ω since conditions (1) and (2) provide the inducing function $\tilde{f}$. By Lemma 2.7 we may slide f to some function $\hat{f}$ of uniform cofinality ω with $[f] = [\hat{f}]$. ∎

2.4 The ω-cofinal, Normal Measure on ω_2

Definition 2.17 *A set D is said to be ω-closed if for any countable increasing sequence $\{\eta_\beta\}_{\beta<\alpha<\omega_1}$ of elements of D, $\sup_{\beta<\alpha}\eta_\beta$ is also in D.*

Definition 2.18 (S_1^1) *We define $S_1^1 \subset \mathcal{P}(\omega_2)$ by*

$$A \in S_1^1 \iff \exists \text{ an } \omega\text{-closed, unbounded set } D \quad D \subset A.$$

It turns out that S_1^1 is, in fact, a normal measure on ω_2. Also, since any ordinal $\eta < \omega_2$ may be represented by a function $f : \omega_1 \to \omega_1$, we have the following characterization of S_1^1.

Lemma 2.19 *Let C be a c.u.b. subset of ω_1 and let*

$$D_C = \{[f] \mid f : \omega_1 \to C \text{ is of c.t. }\}.$$

Then D_C is an ω-closed unbounded subset of ω_2

Proof. Fix $\eta \in \omega_2$ and let $g : \omega_1 \to \omega_1$ represent η. We will proceed by constructing a function $f : \omega_1 \to C$ of correct type with $[f] > \eta$. To do this we will simultaneously construct a function $\hat{f} : \omega \times \omega_1$ which induces f. For $\gamma \in \omega_1$ we define $N_{c,n}(\gamma)$ to be the nth element of C after γ. Then define

$$\hat{f}(n,0) = N_{c,n}(g(0)) \text{ and } f(0) = \sup_{n\in\omega} \hat{f}(n,0).$$

For $\alpha > 0$ define

$$\hat{f}(n,\alpha) = N_{c,n}\left(\max(\sup_{\alpha'<\alpha}(f(\alpha')), g(\alpha))\right)$$

$$\text{and} \qquad f(\alpha) = \sup_{n\in\omega} \hat{f}(n,\alpha).$$

We then observe the following concerning f:

1. $f : \omega_1 \to C$. This is clear by the closure of the set C.

2. f is totally discontinuous. Indeed

$$f(\alpha) = \sup_{n\in\omega} \hat{f}(n,\alpha) > \hat{f}(1,\alpha) > \sup_{\alpha'<\alpha} f(\alpha').$$

3. f is increasing by 2 above.

4. f is of correct type since f is induced by $\hat{f}$.

5. $[f] > [g]$ by construction.

Therefore, f is the desired element of D_C and hence D_C is unbounded in ω_2.

Suppose that $\{[f_i]\}_{i\in\omega}$ is an increasing sequence of elements of D_C. Fix f such that $[f] = \sup_{i\in\omega}[f_i]$. Then $[f]$ is an ordinal of cofinality ω and $[f] > \omega_1$. Let C be a c.u.b. set which witnesses the following:

1. $f \restriction C$ is of uniform cofinality ω. This may be accomplished as in Lemma 2.16.

2. $\forall \beta \in C \;\; f(\beta) > \sup_{\alpha \in C, \alpha < \beta} f(\alpha)$. This condition is guaranteed since otherwise f would be continuous on C and consequently $[f] = \omega_1$, a contradiction.

By Lemma 2.7 we may slide f to some $\hat{f} : \omega_1 \to C$ so that $[f] = [\hat{f}]$, $\hat{f}$ is totally discontinuous (consequently increasing) and $\hat{f}$ is of uniform cofinality ω. Thus, D_C is ω-closed. ∎

2.5 The Supercompactness Measures on $P_\kappa(\lambda)$

Definition 2.20 *Let κ and λ be ordinals. Define $P_\kappa(\lambda)$ to be the set,*

$$\{S \mid S \subset \lambda \text{ and } |S| < \kappa\}.$$

Then a supercompactness measure on $P_\kappa(\lambda)$ is defined to be an ultrafilter μ on $P_\kappa(\lambda)$ satisfying the following:

1. *$\forall \gamma < \lambda \;\; \{S \mid \gamma \in S\} \in \mu$. That is, μ is* fine.

2. *If $F : P_\kappa(\lambda) \to \lambda$ is a function such that $\{S \mid F(S) \in S\} \in \mu$ then there is some $\gamma < \lambda$ such that $\{S \mid F(S) = \gamma\} \in \mu$. That is, μ is* normal *and simply put; every a.e. (almost everywhere) pressing down function is a.e. constant.*

3. *The measure is κ-*additive*. That is, if $\delta < \kappa$ is fixed and for each $\gamma < \delta \;\; \mathcal{A}_\gamma \in \mu$ then $\bigcap_{\gamma < \delta} \mathcal{A}_\gamma \in \mu$.*

This paper will focus on the supercompactness measures obtained on $P_{\omega_1}(\omega_n)$. Then the ω_1-additivity will be referred to as *countable* additivity. A significant result concerning such measures is that, granting AD, supercompactness measures for

$P_{\omega_1}(\omega_n)$ do, in fact, exist for each $n \in \omega$. It should be pointed out here that the supercompactness measure may not necessarily be obtained by doing the obvious thing and playing a *simple* game on ordinals. Such games, in general, are not determined. In fact, in chapter 2 it will be demonstrated that the naïve approach will not work for the supercompactness measure on $P_{\omega_1}(\omega_2)$. A supercompactness measure does, nevertheless, exist for $P_{\omega_1}(\omega_2)$ and indeed for $P_{\omega_1}(\kappa)$ where $\kappa < \boldsymbol{\delta}^2_1$. This can be shown by appealing to a Harrington-Kechris type argument [HK] or using Generic Codes [KW]. The idea behind both approaches is to consider the guaranteed supercompactness measure for some $P_{\omega_1}(\lambda)$ with $\lambda > \kappa$ and then project that measure onto $P_{\omega_1}(\kappa)$. The following proposition will demonstrate that supercompactness measures may be obtained in a manner similar to that just described.

Proposition 2.21 *Suppose μ is the supercompactness measure for $P_{\omega_1}(\lambda)$ and that $\kappa < \lambda$. For $\mathcal{A} \subset P_{\omega_1}(\lambda)$ define*

$$\mathcal{A}_\kappa = \{S \cap \kappa \mid S \in \mathcal{A}\} \text{ and } \mu_\kappa = \{\mathcal{A}_\kappa \mid \mathcal{A} \in \mu\}.$$

Then μ_κ is the supercompactness measure on $P_{\omega_1}(\kappa)$.

Proof. We must first demonstrate that μ_κ is a measure. Let $\mathcal{B} \subset P_{\omega_1}(\kappa)$ be such that $\mathcal{B} \notin \mu_\kappa$. Then

$$\mathcal{A} = \{S \mid S \cap \kappa \in \mathcal{B}\} \notin \mu.$$

Thus $\mathcal{A}^c \in \mu$ and for any $S \in \mathcal{A}^c$ it is easily the case that $S \cap \kappa \notin \mathcal{B}$ which implies that $\mathcal{A}^c_\kappa \subset \mathcal{B}^c$ and so $\mathcal{B}^c \in \mu_\kappa$.

To show that μ_κ is fine, fix any $\eta < \kappa$ and observe that

$$\mathcal{A} = \{S \mid \eta \in S\} \in \mu$$

and

$$\{T \subset P_{\omega_1}(\kappa) \mid \eta \in T\} = \mathcal{A}_\kappa.$$

To show that this measure is normal let $H : P_{\omega_1}(\kappa) \to \kappa$ be pressing down on a set $\mathcal{B} \in \mu_\kappa$. We define an auxiliary function $\hat{H} : P_{\omega_1}(\lambda) \to \kappa$ by $\hat{H}(S) = H(S \cap \kappa)$. Then $\hat{H}$ is pressing down on some set $\mathcal{A} \in \mu$. In fact $\mathcal{A}$ is a measure 1 set inducing $\mathcal{B}$. By the normality of μ there is some $\mathcal{A}'$ so that $\hat{H}$ is constant on $\mathcal{A}'$. So H is constant on $\mathcal{A}'_\kappa$. Thus μ_κ is normal.

The additivity of μ_κ is an easy consequence of the additivity of μ. Indeed, if $\{\mathcal{B}^\beta \mid \beta < \alpha < \omega_1\}$ is a collection of measure 1 sets with respect to μ_κ then there exists a set $\{\mathcal{A}^\beta \mid \beta < \alpha < \omega_1\}$, of sets in μ, which induce them. Then $\bigcap_{\beta<\alpha} \mathcal{B}^\beta$ is induced by $\bigcap_{\beta<\alpha} \mathcal{A}^\beta$ since for any $S \in \bigcap_{\beta<\alpha} \mathcal{A}^\beta \quad S \in \mathcal{A}^\beta$ for all $\beta < \alpha$ which implies that $(S \cap \kappa) \in \mathcal{B}^\beta$ for all $\beta < \alpha$ and thus $(S \cap \kappa) \in \bigcap_{\beta<\alpha} \mathcal{B}^\beta$. ∎

It may have been noticed by the reader that in the previous proposition, the article, *the* was assigned to the supercompactness measure. The usage of this word is justified in the sense that for a given set $P_{\omega_1}(\kappa)$ supporting a supercompactness measure, such a measure is unique. This result is due to Woodin and will be referred to as Woodin's Theorem. For a full discussion see [Wo].

A useful notational convention which will be employed throughout the paper is to let, "$\forall^*_\mu S \;\; P(S)$", stand for the statement, "There is some $\mathcal{A}$ in μ so that for all $S \in \mathcal{A}$, the property, P holds at S." Another way of saying this is, "For almost all $S \;\; P(S)$." Similarly, "$\forall^*_{W^1_1} \alpha \;\; P(\alpha)$", will abbreviate the statement, "There is a c.u.b. set C so that for each α in C, the property P holds at α".

A simple observation concerning the supercompactness measure on $P_{\omega_1}(\omega_1)$ is given by the following

Lemma 2.22 *The supercompactness measure* μ_1 *on* $P_{\omega_1}(\omega_1)$ *is given by the measure* W_1^1 *on* ω_1*.*

Proof. We proceed by first showing that

$$\forall^*_{\mu_1} S \quad S \text{ is an ordinal.}$$

Suppose that this were not true. Then for almost all S there is a least $\gamma_S < \sup(S)$ such that $\gamma_S \notin S$. For each such S we define α_S to be the least element of S greater than γ_S. By normality, there is some α_0 so that

$$\forall^*_{\mu_1} S \quad \alpha_0 \in S \text{ and } \exists \gamma_S < \alpha_0, \gamma_S \notin S.$$

This contradicts the fact that by fineness and countable additivity, the set α_0 is a subset of S for almost all S.

Armed with this observation we may now show that if C is a c.u.b. subset of ω_1 then $C \in \mu_1$. For if this were not true then there is some set $\mathcal{A} \in \mu_1$ so that $\mathcal{A} \cap C = \phi$ and $\mathcal{A}$ satisfies the first observation above. Then for any $S \in \mathcal{A}$, there is a greatest element β_S in $C \cap S$. By the normality of μ_1 there is some β_0 so that $\forall^*_{\mu_1} S \quad \beta_0$ is the greatest element of C which is also in S. But this cannot be since one may take some $\gamma \in C$ so that $\gamma > \beta_0$ and by fineness, argue that $\forall^*_{\mu_1} S \in \mathcal{A} \quad \gamma \in S$.

Thus if $A \in W_1^1$ then there is some c.u.b. set C so that $C \subset A$ and $C \in \mu_1$ which implies that $A \in \mu_1$. Conversely, if $B \notin W_1^1$ then $B^c \in W_1^1$ which implies that $B^c \in \mu_1$ thus $B \notin \mu_1$. ∎

Since all of the sets $S \in P_{\omega_1}(\omega_n)$ are countable sets, there is a natural association between such a set and an ordinal less than ω_1. This association is given by the following

Definition 2.23 *Let S be a countable set. Then the order type of S (abbreviated $o.t.(S)$) is the transitive collapse of S.*

There are several results connecting order types and the supercompactness measures. The first, and perhaps most important of these is given in this theorem due to Jackson.

Theorem 2.24 (Jackson) *Suppose that λ is a cardinal number and C is a c.u.b. subset of ω_1. Let μ be the supercompactness measure on $P_{\omega_1}(\lambda)$. Then*

$$\forall^*_\mu S \;\; o.t.(S) \in C.$$

Proof. The proof will be divided into two cases; the case where λ is a successor cardinal and the limit case.

Case 1 $\lambda = \kappa^+$

Fix C, a c.u.b. subset of ω_1. Suppose, towards a contradiction, that $\forall^*_\mu S \;\; o.t.(S) \notin C$. Fix some $\alpha \in C$ and notice that $\forall^*_\mu S \;\; o.t.(S) > \alpha$. Thus $\forall^*_\mu S$ there is a largest element η of S so that when S is collapsed, η is collapsed to an element of C. This is easily seen by considering an S such that $o.t.(S) \notin C$ and so that $\exists \alpha \in C \;\; \alpha < o.t.(S)$, and observing that $\beta = \sup\{\alpha \in C \mid \alpha < o.t.(S)\} \in C$ then simply consider the element of S which collapses to β.

Using the normality of μ, $\exists\delta < \lambda, \forall^*_\mu S \;\; \delta$ is the largest element of S which collapses to an element of C. At this point, it is likely that as the choice of S varies, so does the element of C to which δ maps.

Define the function $f : \omega_1 \to \omega_1$ by $f(\alpha) = N_C(\alpha)$. Where $N_C(\alpha)$ is the least element in the set $\{\gamma \in C \mid \gamma > \alpha\}$. Using Kunen's argument (see Lemma 2.12) there is a well ordering $\prec$ of ω_1 such that

$$\forall^*_{W^1_1}\alpha \;\; f(\alpha) < |{\prec}\restriction \alpha| \,.$$

This induces an ordering $\ll$ on $[\kappa, \delta)$ as follows:

$$\eta_1 \ll \eta_2 \iff \forall^*_\mu S \;\; \pi_S(\eta_1) \prec \pi_S(\eta_2).$$

where π_S is the transitive collapse mapping from S to $o.t.(S)$.

It must be shown that $\ll$ is, in fact, a well ordering. Supposing, towards a contradiction, that this were not the case we may fix a decreasing sequence

$$\eta_0 \gg \eta_1 \gg \eta_2 \gg \cdots$$

of elements of $[\kappa, \delta)$. Then

$$\begin{aligned} & \forall^*_\mu S \;\; \pi_S(\eta_0) \succ \pi_S(\eta_1) \\ \text{and} \quad & \forall^*_\mu S \;\; \pi_S(\eta_1) \succ \pi_S(\eta_2) \\ \text{and} \quad & \forall^*_\mu S \;\; \pi_S(\eta_2) \succ \pi_S(\eta_3) \\ \text{and} \quad & \vdots \end{aligned}$$

Using the countable additivity of μ we have that

$$\forall^*_\mu S \;\; \pi_S(\eta_0) \succ \pi_S(\eta_1) \succ \pi_S(\eta_2) \succ \cdots$$

This, of course, is impossible. So $\ll$ is a well ordering.

Let $\rho : [\kappa, \delta) \to [\delta, \delta')$ be an order preserving bijection from $([\kappa, \delta), \ll)$ to $([\delta, \delta'), \in)$ where δ' is chosen sufficiently large to work. By the choice of δ, it is the case that

$$\forall^*_\mu S \quad \pi_S(\delta) \text{ is the greatest element of } C < o.t.(S).$$

Also, by countable additivity

$$\forall^*_\mu S \quad \forall \eta_1, \eta_2 \in (S \cap [\kappa, \delta)) \quad (\eta_1 \ll \eta_2 \iff \pi_S(\eta_1) \prec \pi_S(\eta_2)).$$

Finally

$$\forall^*_\mu S \quad \forall \eta \in S \cap [\kappa . \delta) \quad \rho(\eta) \in S.$$

This last statement is a consequence of the normality of μ. In particular, If it were not true, then for each S in some measure 1 set one could find a least η_S such that $\rho(\eta_S) \notin S$. Normality would then yield an η so that $\forall^*_\mu S \quad \rho(\eta) \notin S$. The fineness of μ, however, guarantees that $\forall^*_\mu S \quad \rho(\eta) \in S$, which is impossible.

Using the above three statements,

$$\begin{aligned} \forall^*_\mu S \quad o.t.(S \cap [\delta, \delta')) &\geq |{\ll}\restriction (S \cap [\kappa, \delta))| \\ &= |{\prec}\restriction \pi_S(\delta)| \\ &> f(\pi_S(\delta)) \\ &= N_C(\pi_S(\delta)) \end{aligned}$$

So $\forall^*_\mu S \quad o.t.(S) > N_C(\pi_S(\delta))$ which violates the supposition that $\pi_S(\delta)$ was the greatest element of C less than $o.t.(S)$.

Therefore, $\forall^*_\mu S \quad o.t.(S) \in C$.

Case 2 λ *is a limit cardinal.*

By Proposition 2.21 we have that for any successor cardinal $\kappa < \lambda$, μ_κ is the supercompactness measure for $P_{\omega_1}(\kappa)$. Thus, using the result from Case 1 above and the definition of μ_κ,

$$\forall^*_\mu S \quad o.t.(S \cap \kappa) \in C. \tag{2.1}$$

It must now be shown that for a fixed c.u.b. set C there is some $\mathcal{A} \in \mu$ so that

$$\forall S \in \mathcal{A} \;\; \forall \kappa < \lambda \;\; (S \cap [\kappa^-, \kappa) \neq \phi \Rightarrow o.t.(S \cap \kappa) \in C).$$

Suppose, towards a contradiction, that this were not the case. Then

$$\forall^*_\mu S \;\; \exists \kappa < \lambda \;\; (S \cap [\kappa^-, \kappa) \neq \phi \text{ and } o.t.(S \cap \kappa) \notin C).$$

For any such S there is a least successor cardinal κ_S so that

$$S \cap [\kappa_S^-, \kappa_S) \neq \phi \text{ and } o.t.(S \cap \kappa_S) \notin C.$$

Let η_S be the least element in a set S so that $\eta_S \in [\kappa_S^-, \kappa_S)$. By the normality of μ there is some η_0 which works for almost all S. Let κ_0 be the successor cardinal so that $\eta_0 \in [\kappa_0^-, \kappa_0)$. By the definition of η_0,

$$\forall^*_\mu S \quad o.t.(S \cap \kappa_0) \notin C$$

but this violates 2.1 above.

Let $\mathcal{A} \in \mu$ be a set so that

$$\forall S \in \mathcal{A} \;\; \forall \kappa < \lambda \;\; (S \cap [\kappa^-, \kappa) \neq \phi \Rightarrow o.t.(S \cap \kappa) \in C).$$

Fix $S \in \mathcal{A}$. Let

$$A = \{\kappa \mid \kappa \text{ is a successor cardinal and } S \cap [\kappa^-, \kappa) \neq \phi\}.$$

Since S is countable, the set A is also countable. We can then fix a countable, cofinal sequence $\{\kappa_\beta\}_{\beta<\alpha}$ which is increasing into $\sup A$. Notice that

$$\beta_1 < \beta_2 < \alpha \Rightarrow o.t.(S \cap \kappa_{\beta_1}) \leq o.t.(S \cap \kappa_{\beta_2}).$$

This follows from the fact that $S \cap \kappa_{\beta_1} \subset S \cap \kappa_{\beta_2}$. Letting $\gamma_\beta = o.t.(S \cap \kappa_\beta)$ we have that $\forall \beta < \alpha \;\; \gamma_\beta \in C$ and so by the closure of C, $\sup_{\beta<\alpha} \gamma_\beta \in C$.

What remains, is to show that $o.t.(S) = \sup_{\beta<\alpha} \gamma_\beta$. Let $\delta < o.t.(S)$. Then there is some $\xi \in S$ so that ξ collapses to δ. Let κ_β be the least element in the sequence so that $\xi \in [\kappa_\beta^-, \kappa_\beta)$. Then $\gamma_\beta \geq \delta$ so $\sup_{\beta<\alpha} \gamma_\beta \geq o.t.(S)$. Easily, $o.t.(S) \geq \gamma_\beta$ for any $\beta < \alpha$ since $S \cap \kappa_\beta \subset S$.

Therefore, $o.t.(S) \in C$. ∎

A simple observation which may be made at this point is the related fact that

$$\forall^*_\mu S \;\; (S \cap \omega_1) \in C.$$

This follows from the fact that μ_1, the supercompactness measure on $P_{\omega_1}(\omega_1)$, is the set $\{\mathcal{A}_{\omega_1} \mid \mathcal{A} \in \mu\}$ (as in Proposition 2.21) and from the fact that $\forall^*_{\mu_1} S \;\; S$ is an ordinal (see Lemma 2.22).

Before presenting a final related result, the following simple, yet useful lemma will be given.

Lemma 2.25 *Let μ be the supercompactness measure on $P_{\omega_1}(\lambda)$ and $f : \lambda \to \lambda$ a function. Then μ is* closed *under f. That is*

$$\forall^*_\mu S \quad (\xi \in S \Rightarrow f(\xi) \in S).$$

Proof. Suppose that μ were not closed under f. Then for almost all S there is a least ξ_S so that $\xi_S \in S$ and $f(\xi_S) \notin S$. By the normality of μ, there is some ξ_0 so that

$$\forall^*_\mu S \quad \xi_0 \in S \text{ and } f(\xi_0) \notin S.$$

The fineness of μ, however, guarantees that $\forall^*_\mu S \quad f(\xi_0) \in S$ which is a contradiction.

∎

Lemma 2.26 *Let μ be the supercompactness measure on $P_{\omega_1}(\lambda)$ and let $\kappa < \lambda$ be a successor cardinal. Then*

$$\forall^*_\mu S \quad o.t.(S \cap \kappa) = o.t.(S \cap [\kappa^-, \kappa)).$$

Proof. Certainly, it is the case that

$$\forall^*_\mu S \quad o.t.(S \cap \kappa) \geq o.t.(S \cap [\kappa^-, \kappa))$$

since $(S \cap [\kappa^-, \kappa)) \subset (S \cap \kappa)$. Now suppose, towards a contradiction, that

$$\forall^*_\mu S \quad o.t.(S \cap \kappa) > o.t.(S \cap [\kappa^-, \kappa)).$$

Then for any such S one can fix the least $\eta_S \in S$ so that

$$o.t.(\eta_S \cap S) = o.t.(S \cap [\kappa^-, \kappa)).$$

The normality of μ guarantees that there is some η_0 so that

$$\forall^*_\mu S \quad o.t.(\eta_0 \cap S) = o.t.(S \cap [\kappa^-, \kappa)).$$

Since $\eta_0 < \kappa$, there is some $\delta < \kappa$ and $f : \eta_0 \to [\kappa^-, \delta)$ which is order preserving. By Lemma 2.25, μ is closed under f. Also, by the fineness of μ, $\forall^*_\mu S \quad \delta \in S$. Thus

$$\forall^*_\mu S \quad o.t.(S \cap \eta_0) < o.t.(S \cap [\kappa^-, \kappa))$$

which is, of course, a contradiction. ∎

CHAPTER 3

THE SUPERCOMPACTNESS MEASURE ON $P_{\omega_1}(\omega_2)$

3.1 Characteristics of the Supercompactness Measure on $P_{\omega_1}(\omega_2)$

In this section we consider some of the structural characteristics of μ_2 as well as methods for obtaining μ_2. As indicated in the first chapter, simply playing an ordinal game will not generate the supercompactness measure. Indeed, suppose that $\mathcal{A} \subset P_{\omega_1}(\omega_2)$ and define the game $G^{\mathcal{A}}$ by

$$\begin{array}{c|ccccccc} \mathrm{I} & \eta_0 & & \eta_2 & & \eta_4 & \cdots \\ \mathrm{II} & & \eta_1 & & \eta_3 & & \cdots \end{array}$$

where two players alternately play ordinals less than ω_2 to produce a countable set S. Player II wins if and only if $S \in \mathcal{A}$

Proposition 3.1 *There is some set $\mathcal{A}$ so that the game, $G^{\mathcal{A}}$ is not determined.*

Proof. Let $\mathcal{A}$ be the set of all $S \in P_{\omega_1}(\omega_2)$ so that $S \cap \omega_1$ is an ordinal. Suppose that player II had a winning strategy in the game $G^{\mathcal{A}}$. The problem here is that player I could play any ordinal, $\beta < \omega_1$ and player II would have the responsibility of completely enumerating some ordinal α such that $\beta \leq \alpha < \omega_1$. Restricting this enumeration to β yields an enumeration of β. So player II's strategy provides an enumeration for any ordinal less than ω_1. This violates the Axiom of Determinacy. Therefore, player II does not have a winning strategy.

Suppose that player I had a winning strategy in $G^{\mathcal{A}}$. We will proceed to show that player II can, in fact, beat player I's strategy by producing a set S in $\mathcal{A}$. Suppose player I's first move is η_0. If $\eta_0 \geq \omega_1$ then player II will simply copy player I's move and play η_0 as well. If, however, $\eta_0 < \omega_1$, then player II will fix an enumeration of η_0 and begin by playing the first element of that enumeration.

Similarly, if player I plays $\eta_2 \geq \omega_1$ then player two will continue with his enumeration of η_0. Otherwise, player II will fix an enumeration of η_2 and alternate playing elements of the two enumerations.

In general, if at any time player I plays an ordinal less than ω_1 then player II will simply fix an enumeration of that ordinal and continue alternating plays between elements of all of the enumerations. Then the set S produced by these two players will, in fact, be an element of $\mathcal{A}$ and so player II will have beat player I's winning strategy. Thus player I does not have a winning strategy in this game.

Therefore, this game is not determined. $\blacksquare$

Notice that there was nothing unique to the supercompactness measure on $P_{\omega_1}(\omega_2)$ employed in this proof. So what has really been shown is that no ordinal game of this nature is determined. This causes no real problems since as stated previously, the supercompactness measures do exist. Often, however, generating such a measure requires the use of some less intuitive means e.g. a Harrington-Kechris type argument or the use of Generic Codes. There are, of course, exceptions. Lemma 2.22 yields the supercompactness measure for $P_{\omega_1}(\omega_1)$ using only the measure W_1^1, and the following discussion provides a method for obtaining μ_2.

For any $\eta < \omega_2$ a bijection $\rho : \omega_1 \rightarrow \eta$ can be fixed. Notice then that for any

$\alpha < \omega_1 \quad \rho''(\alpha) = \{\rho(\beta) \mid \beta < \alpha\} \in P_{\omega_1}(\eta)$.

Lemma 3.2 *Let ρ_1 and ρ_2 be two bijections from ω_1 onto η. Then*

$$\forall^*_{W_1^1}\alpha \quad \rho_1''(\alpha) = \rho_2''(\alpha).$$

Proof. Suppose, towards a contradiction, that $\forall^*_{W_1^1}\alpha \quad \rho_1''(\alpha) \neq \rho_2''(\alpha)$. Then for each such α it is the case that either there is some $\beta < \alpha$ so that $\rho_1(\beta) \notin \rho_2''(\alpha)$ or there is some $\beta < \alpha$ so that $\rho_2(\beta) \notin \rho_1''(\alpha)$ Without loss of generality we may assume the former. Let β_α be the least such β for each α. Using the normality of W_1^1, we have that there is some β so that $\forall^*_{W_1^1}\alpha \quad \rho_1(\beta) \notin \rho_2''(\alpha)$. Fix $\gamma < \omega_1$ so that $\rho_2(\gamma) = \rho_1(\beta)$. Notice that $\{\alpha \mid \alpha > \gamma \text{ and } \alpha > \beta\} \in W_1^1$. So $\forall^*_{W_1^1}\alpha \quad \rho_2(\gamma) \in \rho_2''(\alpha)$. Thus $\forall^*_{W_1^1}\alpha \quad \rho_1(\beta) \in \rho_2''(\alpha)$, a contradiction. ∎

This invariance with respect to ρ yields the following observation. Suppose that for some fixed $\eta < \omega_2$ there is a relation $P \subset p_{\omega_1}(\eta) \times \omega_1$. Then for any two bijections $\rho_1, \rho_2 : \omega_1 \to \eta$ we have that

$$\forall^*_{W_1^1}\alpha \quad P(\rho_1''(\alpha), \alpha) \iff P(\rho_2''(\alpha), \alpha).$$

That is, in cases where there seems to be a reliance on the choice of the bijection, that reliance may be suppressed. This observation will be of some importance throughout this paper.

Lemma 3.3 *Let $\mathcal{A} \subset P_{\omega_1}(\omega_2)$. Then*

$$\mathcal{A} \in \mu_2 \iff (\forall^*_{S_1^1}\eta \quad \exists\rho : \omega_1 \to \eta \quad \forall^*_{W_1^1}\alpha \quad S_{\eta,\rho,\alpha} \in \mathcal{A}).$$

Here $S_{\eta,\rho,\alpha} = \rho''(\alpha)$.

Proof. Notice, first, if we fix an η such that

$$\exists \rho : \omega_1 \to \eta \ \ \forall^*_{W^1_1}\alpha \ \ S_{\eta,\rho,\alpha} \in \mathcal{A}$$

then for any other bijection ρ' it is also the case that $\forall^*_{W^1_1}\alpha \ \ S_{\eta,\rho',\alpha} \in \mathcal{A}$ by the remarks above. We may then suppress the choice of the bijection and consider the set $S_{\eta,\alpha}$ with the understanding that a bijection has been fixed. Let

$$m = \{\mathcal{A} \mid \forall^*_{S^1_1}\eta \ \ \forall^*_{W^1_1}\alpha \ \ S_{\eta,\alpha} \in \mathcal{A}\}.$$

Observe that m is a measure on $P_{\omega_1}(\omega_2)$ since if $\mathcal{A} \notin m$ then easily

$$\forall^*_{S^1_1}\eta \ \ \forall^*_{W^1_1}\alpha \ \ S_{\eta,\alpha} \in \mathcal{A}^c.$$

To show that m is fine fix $\xi \in \omega_2$. Then the set

$$\{\eta \mid \eta > \xi\} \in S^1_1.$$

Fix such an η and a bijection, $\rho : \omega_1 \to \eta$. Then there is some $\beta < \omega_1$ so that $\rho(\beta) = \xi$. The set

$$\{\alpha \mid \alpha > \beta\} \in W^1_1$$

and for any such α,

$$\xi = \rho(\beta) \in \rho''(\alpha) = S_{\eta,\alpha}.$$

Thus, letting $\mathcal{A} = \{S \mid \xi \in S\}$ it is the case that

$$\forall^*_{S^1_1}\eta \ \ \forall^*_{W^1_1}\alpha \ \ S_{\eta,\alpha} \in \mathcal{A}.$$

To show that m is normal we let $H : P_{\omega_1}(\omega_2) \to \omega_2$ be an a.e. pressing down function with respect to m. That is

$$\forall^*_{S^1_1}\eta \;\; \forall^*_{W^1_1}\alpha \;\; H(S_{\eta,\alpha}) \in S_{\eta,\alpha}.$$

Fix such an η and notice that

$$\forall^*_{W^1_1}\alpha \;\; \exists\beta^\eta_\alpha < \alpha \;\; H(S_{\eta,\alpha}) = \rho(\beta^\eta_\alpha).$$

By the normality of W^1_1, there is some β^η so that $\forall^*_{W^1_1}\alpha \;\; H(S_{\eta,\alpha}) = \rho(\beta^\eta)$. Let $\gamma^\eta = \rho(\beta^\eta)$. Then by the normality of S^1_1, there is some γ so that

$$\forall^*_{S^1_1}\eta \;\; \forall^*_{W^1_1}\alpha \;\; H(S_{\eta,\alpha}) = \gamma.$$

Thus m is normal.

The additivity of the measure m comes from the fact that both S^1_1 and W^1_1 are countably additive. Specifically, let $\{\mathcal{A}_n\}_{n\in\omega}$ be a sequence of elements of m. So

$$\forall n \in \omega \;\; \exists D_n \;\; \forall \eta \in D_n \;\; \forall^*_{W^1_1}\alpha \;\; S_{\eta,\alpha} \in \mathcal{A}_n$$

where $D_n \in S^1_1$ for each n. Let $D = \bigcap_{n\in\omega} D_n$ and fix $\eta \in D$. Since for each $n, \eta \in D_n$ then

$$\forall^*_{W^1_1}\alpha \;\; S_{\eta,\alpha} \in \mathcal{A}_n.$$

That is, for each n there is a c.u.b. set C_n and a bijection $\rho_n : \omega_1 \to \eta$ so that

$$\forall \alpha \in C_n \;\; S_{\eta,\rho_n,\alpha} \in \mathcal{A}_n.$$

Now for any pair of bijections, (ρ_i, ρ_j), there is a c.u.b. set, $C_{i,j}$ so that

$$\forall \alpha \in C_{i,j} \;\; S_{\eta,\rho_i,\alpha} = S_{\eta,\rho_j,\alpha}.$$

Letting

$$C = (\bigcap_{n\in\omega} C_n) \cap (\bigcap_{(i,j)\in\omega\times\omega} C_{i,j})$$

we have that $\forall\alpha \in C \;\; S_{\eta,\alpha} \in \bigcap_{n\in\omega} \mathcal{A}_n$. So m is countably additive.

By Woodin's Theorem, $m = \mu_2$. ∎

There are several consequences of this characterization of μ_2. The first is that since a c.u.b. subset C of ω_1 can be used to generate D_C, an ω-closed unbounded subset of ω_2 (see Lemma 2.19), then μ_2 may be characterized as follows: $\mathcal{A} \in \mu_2 \iff$

$$\exists C \;\; \forall f : \omega_1 \to C \text{ of c.t. } \; \exists \rho : \omega_1 \to [f] \;\; \forall^*_{W^1_1}\alpha \;\; S_{f,\rho,\alpha} \in \mathcal{A}.$$

A second consequence is the following simple lemma.

Lemma 3.4

$$\forall^*_{S^1_1}\eta \;\; \forall^*_{W^1_1}\alpha \;\; (S_{\eta,\alpha} \cap \omega_1) = \alpha$$

Proof. Assume to the contrary. By Lemmas 2.21 and 2.22 we have that there is some $\mathcal{A} \in \mu_2$ so that $\forall S \in \mathcal{A} \;\; S \cap \omega_1$ is an ordinal. Since $\mathcal{A} \in \mu_2$,

$$\forall^*_{S^1_1}\eta \;\; \forall^*_{W^1_1}\alpha \;\; S_{\eta,\alpha} \in \mathcal{A}.$$

Fix such an η and a bijection $\rho : \omega_1 \to \eta$. Letting $\gamma_{\eta,\alpha} = S_{\eta,\rho,\alpha} \cap \omega_1$ we have that either $\forall^*_{W^1_1}\alpha \;\; \gamma_{\eta,\alpha} < \alpha$ or $\forall^*_{W^1_1}\alpha \;\; \gamma_{\eta,\alpha} > \alpha$ by our assumption.

Consider the case where

$$\forall^*_{S^1_1}\eta \;\; \forall^*_{W^1_1}\alpha \;\; \gamma_{\eta,\alpha} < \alpha.$$

Fix such an η and notice that by the normality of W^1_1, there is some γ_η so that $\forall^*_{W^1_1}\alpha \;\; S_{\eta,\alpha} \cap \omega_1 = \gamma_\eta$. This is easily false.

Consider the case where

$$\forall^*_{S^1_1}\eta \ \ \forall^*_{W^1_1}\alpha \ \ \gamma_{\eta,\alpha} > \alpha.$$

Again, fixing an η satisfying the above assumption and a bijection ρ we have that for each α there is some $\beta_\alpha < \alpha$ so that $\rho(\beta_\alpha) = \alpha$. By normality there is some β_0 so that $\forall^*_{W^1_1}\alpha \ \ \rho(\beta_0) = \alpha$ but this, too, is absurd.

Thus

$$\forall^*_{S^1_1}\eta \ \ \forall^*_{W^1_1}\alpha \ \ (S_{\eta,\alpha} \cap \omega_1) = \alpha$$

∎

As a final characteristic of μ_2 we present the following

Lemma 3.5 *Let D be an ω-closed unbounded subset of ω_2. Then*

$$\forall^*_{\mu_2}S \quad \sup S \in D.$$

Proof. Suppose that this were not true. Let D be an ω-cofinal unbounded subset of ω_2 so that $\forall^*_{\mu_2}S \ \ \sup S \notin D$. Then clearly $\forall^*_{\mu_2}S \ \ \sup S \notin D'$ where D' is the set of limit points of D where by limit point we mean the limit point of an increasing ω-sequence. Thus

$$\forall^*_{S^1_1}\eta \ \ \forall^*_{W^1_1}\alpha \ \ \sup S_{\eta,\alpha} \notin D'.$$

In fact, since D' is also ω-closed and unbounded, it may be assumed that $\exists E \subset D'$ with $E \in S^1_1$ such that

$$\forall \eta \in E \ \ \forall^*_{W^1_1}\alpha \ \ \sup S_{\eta,\alpha} \notin D'$$

Fix $\eta \in E$. Then since η is the limit point of some increasing ω-sequence, the cofinality of η is ω. Fix a countable cofinal sequence $\{\xi_n\}_{n\in\omega}$ into η. Since $\{\xi_n\}_{n\in\omega}$ is a countable set, there is some $\beta < \omega_1$ so that each element of $\{\xi_n\}_{n\in\omega}$ is in $\rho''(\beta)$. Then the $\sup S_{\eta,\beta} = \eta$. Also, $\forall^*_{W^1_1}\alpha \;\; \alpha > \beta$ thus $\forall^*_{W^1_1}\alpha \;\; \sup S_{\eta,\alpha} = \eta$ and this contradicts our original assumption. ∎

3.2 Order Type Results

An interesting relationship between the functions representing the suprema of the sets in a supercompactness measure and the order type of those sets will be demonstrated by the following

Theorem 3.6 *Let $f : \omega_1 \to \omega_1$ be a function representing some ordinal $\eta < \omega_2$. Fix a bijection $\rho : \omega_1 \to [f]$. Then*

$$\forall^*_{W^1_1}\alpha \;\; o.t.(S_\alpha) = f(\alpha)$$

where $S_\alpha = \rho''(\alpha)$.

Proof. Let f and ρ be as in the hypothesis. By lemma 2.12 there is some well ordering $\prec$ of ω_1 so that

$$\forall^*_{W^1_1}\alpha \;\; f(\alpha) = |\prec\restriction\alpha|.$$

Then, by definition, for each α

$$|\prec\restriction\alpha| = \sup_{\gamma<\alpha} |\gamma|_{\prec\restriction\alpha}.$$

For each $\beta < \omega_1$ there is some $g_\beta : \omega_1 \to \omega_1$ such that $\rho(\beta) = [g_\beta]$. Notice that $[g_\beta] < [f]$. Then

$$\forall^*_{W^1_1}\alpha \;\; g_\beta(\alpha) < |\prec\restriction\alpha|$$

thus there is some $\gamma_\beta \in \omega_1$ so that

$$\forall^*_{W^1_1}\alpha \quad g_\beta(\alpha) = |\gamma_\beta|_{\prec\restriction\alpha}.$$

So there is a bijection $\hat{\rho} : \omega_1 \to \omega_1$ defined by $\hat{\rho}(\beta) = \gamma_\beta$. Fix a c.u.b. set C so that for any $\alpha \in C$, if $\beta < \alpha$ then $\hat{\rho}(\beta) < \alpha$, and so that $f(\alpha) = |\prec\restriction \alpha|$.

Fix $\alpha \in C$ and consider $\beta_1, \beta_2 \in \omega_1$ such that $\rho(\beta_1), \rho(\beta_2) \in S_\alpha$. Then

$$\begin{aligned}
\rho(\beta_1) < \rho(\beta_2) \quad &\iff \quad [g_{\beta_1}] < [g_{\beta_2}] \\
&\iff \quad \forall^*_{W^1_1}\delta \quad g_{\beta_1}(\delta) < g_{\beta_2}(\delta) \\
&\iff \quad \forall^*_{W^1_1}\delta \quad |\gamma_{\beta_1}|_{\prec\restriction\delta} < |\gamma_{\beta_2}|_{\prec\restriction\delta} \\
&\iff \quad |\gamma_{\beta_1}|_\prec < |\gamma_{\beta_2}|_\prec \\
&\iff \quad \gamma_{\beta_1} \prec \gamma_{\beta_2}
\end{aligned}$$

So

$$o.t.(S_\alpha) = \sup_{\gamma<\alpha} |\gamma|_{\prec\restriction\alpha} = |\prec\restriction \alpha| = f(\alpha).$$

∎

CHAPTER 4

THE LOWER BOUND FROM OUTSIDE THE ULTRAPOWER

4.1 Introduction

There are two approaches used in this paper to obtain the lower bound. One is to work from *inside* the ultrapower and the other is, obviously, to work from *outside* the ultrapower. By way of a brief explanation, working inside the ultrapower requires that a set S of ordinals less than ω_2 be represented by a countable set of functions. On a more technical level, the lower bound results make extensive use of Woodin's Theorem. Working outside the ultrapower requires neither a representation of a set by functions nor the use of Woodin's theorem.

By the results given in the previous chapters one may assume that for any c.u.b. set C there is a measure 1 set $\mathcal{A} \in \mu_n$ so that

$$\forall S \in \mathcal{A} \;\; \forall 1 \leq m \leq n \;\; o.t.(S \cap \omega_m) \in C.$$

As before, for any $S \in P_{\omega_1}(\omega_n)$ $\sup(S \cap \omega_2)$ may be represented by a function $\overline{s} : \omega_1 \to \omega_1$. In fact, since μ_2 may be realized as the restriction of μ_n to ω_2, it is the case that there is some C, a c.u.b. set, so that $\forall^*_{\mu_n} S \;\; \sup(S \cap \omega_2)$ may be represented by $\overline{s} : \omega_1 \to C$ of correct type.

4.2 The Lower Bound for $J_{\mu_m}(\omega_n)$

We will proceed to prove the main theorem by first considering the following

Lemma 4.1 $\forall m \geq 1$

$$j_{\mu_m}(\omega_1) \geq \omega_{m+1}$$

Proof. Fix $m \geq 1$ and let θ be an ordinal less than ω_{m+1}. Let $f : \omega_1^m \to \omega_1$ represent θ. To f we associate the function F defined by

$$F(S) = f(\alpha_S^1, \alpha_S^2, \dots, \alpha_S^m).$$

Here $\alpha_S^i = o.t.(S \cap \omega_i)$ for $1 \leq i \leq m$. Assume that $f \sim g$, that F corresponds to f and G corresponds to g. There exists a c.u.b. set, C, such that

$$\forall \vec{\alpha} \in C^m \quad f(\vec{\alpha}) = g(\vec{\alpha}).$$

Since $\forall^*_{\mu_m} S \quad \forall 1 \leq i \leq m \quad \alpha_S^i \in C$, we have that

$$\forall^*_{\mu_m} S \quad F(S) = f(\alpha_S^1, \alpha_S^2, \dots, \alpha_S^m) = g(\alpha_S^1, \alpha_S^2, \dots, \alpha_S^m) = G(S).$$

We may, therefore, define a mapping $\pi : \omega_{m+1} \to j_{\mu_m}(\omega_1)$ by $\pi([f]) = [F]$. By the above remarks, π is well defined. It may likewise be shown that π is order preserving by substituting "$<$" for "$\sim$" and carrying out a similar argument to that above. So we have that π is an embedding. ∎

The main result will now be demonstrated.

Theorem 4.2 $\forall m \geq 2, \forall n \geq 1$

$$j_{\mu_m}(\omega_n) \geq \omega_{2n+m-1}$$

Proof. Fix $m, n \geq 2$ (the $n = 1$ case was shown above) and fix an ordinal $\theta < \omega_{2n+m-1}$. Then θ may be represented by a function $f : \omega_1^{2n+m-2} \to \omega_1$. As usual, for $S \in P_{\omega_1}(\omega_m)$ and $1 \leq i \leq m$ we define $\alpha_S^i = o.t.(S \cap \omega_i)$ and $\overline{s} : \omega_1 \to \omega_1$ as a function representing $\sup(S \cap \omega_2)$.

For any $f : \omega_1^{2n+m-2} \to \omega_1$ we may define F by

$$F(S)(\delta_1, \delta_2, \dots, \delta_{n-1}) = f(\alpha_S^1, \alpha_S^2, \dots, \alpha_S^m, \delta_1, \overline{s}(\delta_1), \dots, \delta_{n-1}, \overline{s}(\delta_{n-1})).$$

The first observation is that the class [F(S)] really only depends on the class $[\overline{s}]$. Indeed, if for a fixed S both $\overline{s}$ and $\overline{t}$ represent $\sup(S \cap \omega_2)$ then $\exists \hat{C} \ \ \forall(\delta_1, \dots, \delta_{n-1}) \in \hat{C}^{n-1}$

$$\begin{aligned} & f(\alpha_S^1, \dots, \alpha_S^m, \delta_1, \overline{s}(\delta_1), \dots, \delta_{n-1}, \overline{s}(\delta_{n-1})) \\ = \quad & f(\alpha_S^1, \dots, \alpha_S^m, \delta_1, \overline{t}(\delta_1), \dots, \delta_{n-1}, \overline{t}(\delta_{n-1})) \end{aligned}$$

by Lemma 2.4.

What must be shown is that if $f \sim g$ with F defined by f and G defined by g as above, then $\forall^*_{\mu_m} S \ \ F(S) \sim G(S)$, that is

$$\forall^*_{\mu_m} S \ \ \forall^*_{W_1^{n-1}} \vec{\delta} \ \ F(S)(\vec{\delta}) = G(S)(\vec{\delta}).$$

Let C be a c.u.b. set so that

$$\forall \vec{\alpha} \in C^{2n+m-2} \ \ f(\vec{\alpha}) = g(\vec{\alpha}).$$

Then $\forall^*_{\mu_m} S \ \ \forall 1 \leq i \leq m \ \ \alpha_S^i \in C$ and $\sup(S \cap \omega_2)$ may be represented by $\overline{s} : \omega_1 \to C$ of c.t.. Fix such an S and let $\overline{s}$ represent $\sup(S \cap \omega_2)$. Then

$\forall(\delta_1, \delta_2, \ldots, \delta_{n-1}) \in C^{n-1}$

$$\begin{aligned}
& F(S)(\delta_1, \delta_2 \ldots, \delta_{n-1}) \\
= \;& f(\alpha_S^1, \alpha_S^2, \ldots, \alpha_S^m, \delta_1, \overline{s}_F(\delta_1), \ldots, \delta_{n-1}, \overline{s}_F(\delta_{n-1})) \\
= \;& g(\alpha_S^1, \alpha_S^2, \ldots, \alpha_S^m, \delta_1, \overline{s}_G(\delta_1), \ldots, \delta_{n-1}, \overline{s}_G(\delta_{n-1})) \\
= \;& G(S)(\delta_1, \delta_2 \ldots, \delta_{n-1})
\end{aligned}$$

Let $\pi : \omega_{2n+m-1} \to j_{\mu_m}(\omega_n)$ be defined by $\pi([f]) = [F]$ as above. Then π is well-defined and similarly order preserving. Thus π is an embedding. ∎

4.3 Related Lower Bound Results

Lemma 4.3 *let κ be a cardinal for which a supercompactness measure μ_κ on $P_{\omega_1}(\kappa)$ exists. Then*

$$j_{\mu_\kappa}(\omega_1) \geq \kappa^+$$

Proof. Fix $\theta < \kappa^+$ and let $\rho : \kappa \to \theta$ be a bijection. Notice that ρ induces a well-ordering $\prec$ of κ with length θ in the following way; for $\gamma_1, \gamma_2 \in \kappa$,

$$\gamma_1 \prec \gamma_2 \iff \rho(\gamma_1) < \rho(\gamma_2).$$

We wish to define a function $F : P_{\omega_1}(\kappa) \to \omega_1$ corresponding to θ. Let F be defined by $F(S) = | \prec\restriction S|$. That is, we consider the length of the well-ordering $\prec$ applied to the ordinals in the set S. Notice that F is a function into ω_1 since each $S \in P_{\omega_1}(\kappa)$ is a countable set.

Suppose that $\lambda < \theta$. Then there is some $\gamma_\lambda < \kappa$ so that $\rho(\gamma_\lambda) = \lambda$. By the fineness of μ_κ, $\forall^* \mu_\kappa S \;\; \gamma_\lambda \in S$. Then letting $|\gamma|_{\prec \restriction S}$ be the length of γ in the well-ordering $\prec$ restricted to elements of S, and defining $F_\lambda : P_{\omega_1}(\kappa) \to \omega_1$ by $F_\lambda(S) = |\gamma_\lambda|_{\prec \restriction S}$, we have that

$$\forall^*_{\mu_\kappa} S \;\; F_\lambda(S) < F(S).$$

So $[F_\lambda] < [F]$.

Fix $\lambda_1, \lambda_2 < \theta$. By the preceding arguments, there are ordinals $\gamma_{\lambda_1}, \gamma_{\lambda_2} < \kappa$ so that $\lambda_1 = \rho(\gamma_{\lambda_1})$ and $\lambda_2 = \rho(\gamma_{\lambda_2})$. Fineness again guarantees that

$$\forall^*_{\mu_\kappa} S \;\; (\gamma_{\lambda_1}, \gamma_{\lambda_2} \in S).$$

Then

$$\begin{aligned}
\lambda_1 < \lambda_2 \quad &\Longleftrightarrow \quad \rho(\gamma_{\lambda_1}) \prec \rho(\gamma_{\lambda_2}) \\
&\Longleftrightarrow \quad \forall^*_{\mu_\kappa} S \;\; |\gamma_{\lambda_1}|_{\prec \restriction S} < |\gamma_{\lambda_2}|_{\prec \restriction S} \\
&\Longleftrightarrow \quad \forall^*_{\mu_\kappa} S \;\; F_{\lambda_1}(S) < F_{\lambda_2}(S) \\
&\Longleftrightarrow \quad [F_{\lambda_1}] < [F_{\lambda_2}]
\end{aligned}$$

So the function F corresponding to the ordinal θ is at least as great as θ. Therefore,

$$j_{\mu_\kappa}(\omega_1) \geq \kappa^+.$$

∎

CHAPTER 5

THE LOWER BOUND FROM INSIDE THE ULTRAPOWER

The aim of this chapter is to provide the lower bound results from inside the ultrapower. Working inside the ultrapower we will represent a set $S \in P_{\omega_1}(\omega_2)$ as a countable collection of functions. We will generally denote this representation by S as well since context will make clear whether we are using a set of ordinals or a set of functions. To simplify notation, we will let $[S]$ be the element of $P_{\omega_1}(\omega_2)$ represented by S. That is, $[S] = \{[s] \mid s \in S\}$. By the countable additivity of W_1^1, if S and T both represent the same set then there is some c.u.b. set C so that

$$\forall s \in S \;\; \exists t \in T \;\; \forall \alpha \in C \;\; s(\alpha) = t(\alpha)$$

and, vice-versa. In this case we say $S \sim T$. For any set of functions S, we define

$$S(\alpha) = \{s(\alpha) \mid s \in S\}.$$

Then if $\prec$ is a well-ordering of ω_1 and $\alpha \in \omega_1$ we define $|\prec\restriction S(\alpha)|$ to be the supremum of the set $S(\alpha)$ ordered by $\prec$. Also, if $g : \omega_1 \to \omega_1$ we define

$$|g(\alpha)|_{\prec\restriction S(\alpha)} = |\prec\restriction \{s(\alpha) \in S(\alpha) \;:\; s(\alpha) \preceq g(\alpha)\}|.$$

5.1 Preliminary Results

Lemma 5.1 *Let $Q \subset \omega_1 \times \omega_1$ be some relation on pairs of ordinals. Suppose that*

$$\forall \alpha \in \omega_1 \;\; \exists C, \text{ a c.u.b. set, such that for all } \beta \in C \;\; Q(\alpha, \beta)$$

then

$$\exists C, \text{ a c.u.b. set, so that } \forall\beta \in C \ \ \forall\alpha < \beta \ \ Q(\alpha, \beta)$$

Proof. Let Q be as in the hypothesis and define a partition P on ordinals in ω_1 by

$$P(\beta) = 1 \iff \forall\alpha < \beta \ \ Q(\alpha, \beta).$$

Let C_0 be homogeneous for P. We will demonstrate that C_0 is homogeneous for the 1-side. Fix $\gamma_0 \in C_0$. Then for each $\alpha < \gamma_0$ there is some C_α so that

$$\forall\beta \in C_\alpha \ \ Q(\alpha, \beta).$$

Let $C_1 = (\bigcap_{\alpha<\gamma_0} C_\alpha) \cap C_0$ and let γ_1 be the least element of C_1 such that $\gamma_1 > \gamma_0$. We may similarly form a c.u.b. set C_2 defined by $C_2 = (\bigcap_{\alpha<\gamma_1} C_\alpha) \cap C_1$ with γ_2 defined to be the least element of C_2 greater than γ_1. We continue this process so that $\forall i \in \omega$, we have defined C_i and γ_i. Let $\gamma = \sup_{i\in\omega} \gamma_i$.

Fix $j \in \omega$ and notice that

$$\forall i \geq j \ \ \gamma_i \in C_j$$

so $\gamma \in C_j$. This is, of course, a consequence of the closure of C_j. Fix $\alpha < \gamma$. Then there exists some γ_i so that $\alpha < \gamma_i$. Thus $\forall\beta \in C_{i+1} \ \ Q(\alpha, \beta)$. In particular, since $\gamma \in C_{i+1}$, $Q(\alpha, \gamma)$. Therefore, γ is an element of C_0 witnessing that C_0 is homogeneous for the 1-side and consequently, C_0 is our desired c.u.b. set. ∎

The following lemmas will demonstrate a sort of invariance with respect to some of the measures we have and will be using.

Lemma 5.2 *Suppose that* $\prec_1$ *and* $\prec_2$ *are well-orderings of* ω_1 *so that* $| \prec_1 | \leq | \prec_2 |$*. Then* $\forall^*_{W^1_1}\gamma \ \ | \prec_1\restriction \gamma| \leq | \prec_2\restriction \gamma|$*.*

Proof. Let $\prec_1, \prec_2$ be as in the hypothesis. Let $f : \omega_1 \to \omega_1$ be the function defined by $f(\alpha) = \beta$ where $|\alpha|_{\prec_1} = |\beta|_{\prec_2}$. By Lemma 2.5 there exists a c.u.b. set C closed under f. That is

$$\forall\gamma \in C \ \ (\alpha < \gamma \Rightarrow f(\alpha) < \gamma).$$

Fix $\gamma \in C$ and let $\alpha, \beta < \gamma$. Then

$$\alpha \prec_1 \beta \Rightarrow f(\alpha) \prec_2 f(\beta)$$

so $\prec_1\restriction \gamma$ may be embedded in $\prec_2\restriction \gamma$ thus $| \prec_1\restriction \gamma| \leq | \prec_2\restriction \gamma|$, and this is true for each $\gamma \in C$. ∎

Lemma 5.3 *Let* $\prec_1$ *and* $\prec_2$ *be well-orderings of* ω_1*. Suppose* δ_1 *and* δ_2 *are elements of* ω_1 *so that* $|\delta_1|_{\prec_1} \leq |\delta_2|_{\prec_2}$*, then* $\forall^*_{W^1_1}\gamma \ \ |\delta_1|_{\prec_1\restriction\gamma} \leq |\delta_2|_{\prec_2\restriction\gamma}$

Proof. Let $\prec_1, \delta_1$ and $\prec_2, \delta_2$ be as in the hypothesis. Let $f : \omega_1 \to \omega_1$ be as in Lemma 5.2. Then for any $\alpha \in \omega_1$,

$$|\alpha|_{\prec_1} = |f(\alpha)|_{\prec_2}. \tag{5.1}$$

Also if $\alpha \prec_1 \delta_1$ then $f(\alpha) \prec_2 \delta_2$. If this were not true then

$$|\alpha|_{\prec_1} < |\delta_1|_{\prec_1} \leq |\delta_2|_{\prec_2} \leq |f(\alpha)|_{\prec_2}$$

which violates equation 5.1. As before, fix a c.u.b. set C so that

$$\forall\gamma \in C \ \ (\alpha < \gamma \Rightarrow f(\alpha) < \gamma).$$

Fix $\gamma \in C$. If $\alpha_1, \alpha_2 < \gamma$ with $\alpha_1 \prec_1 \alpha_2 \prec_1 \delta_1$ then $f(\alpha_1), f(\alpha_2) < \gamma$ and $f(\alpha_1) \prec_2 f(\alpha_2) \prec_2 \delta_2$. Thus, for a fixed $\gamma \in C$, f is an embedding of ordinals $\prec_1 \delta_1$ to ordinals $\prec_2 \delta_2$ and $|\delta_1|_{\prec_1 \restriction \gamma} \leq |\delta_2|_{\prec_2 \restriction \gamma}$. Since this is true for almost all γ, the desired result follows. ∎

Lemma 5.4 *Let $f_1, f_2 : \omega_1 \to \omega_1$ and $\prec_1, \prec_2$ be well-orderings of ω_1 such that*

$$\forall^*_{W_1^2}(\alpha, \beta) \ \ |f_1(\alpha)|_{\prec_1 \restriction \beta} \leq |f_2(\alpha)|_{\prec_2 \restriction \beta}$$

then

$$\forall^*_{W_1^1} \alpha \ \ |f_1(\alpha)|_{\prec_1} \leq |f_2(\alpha)|_{\prec_2}.$$

Proof. Let $f_1, f_2, \prec_1$ and $\prec_2$ be as in the hypothesis. Let C_1 be a c.u.b. set so that

$$\forall (\alpha, \beta) \in C_1^2 \ \ |f_1(\alpha)|_{\prec_1 \restriction \beta} \leq |f_2(\alpha)|_{\prec_2 \restriction \beta}.$$

Suppose, towards a contradiction, that the conclusion were false. Then

$$\exists C_2 \ \ \forall \alpha \in C_2 \ \ |f_1(\alpha)|_{\prec_1} > |f_2(\alpha)|_{\prec_2}.$$

Fix $\alpha \in C_1 \cap C_2$. Then $|f_1(\alpha)|_{\prec_1} > |f_2(\alpha)|_{\prec_2}$.

By Lemma 5.3 we have that there is some C_3 such that

$$\forall \beta \in C_3 \ \ |f_1(\alpha)|_{\prec_1 \restriction \beta} > |f_2(\alpha)|_{\prec_2 \restriction \beta}.$$

Fix $\beta \in C_1 \cap C_3$. Then $(\alpha, \beta) \in C_1^2$ and so

$$|f_1(\alpha)|_{\prec_1 \restriction \beta} \leq |f_2(\alpha)|_{\prec_2 \restriction \beta}$$

On the other hand, $\beta \in C_3$ guarantees that

$$|f_1(\alpha)|_{\prec_1 \restriction \beta} > |f_2(\alpha)|_{\prec_2 \restriction \beta}$$

and this is a contradiction. ∎

Lemma 5.5 *Suppose $\prec_1, \prec_2$ are well-orderings of ω_1 such that $|\prec_1| \leq |\prec_2|$. Then*

$$\forall^*_{\mu_2} S \ \ \forall^*_{W^1_1} \alpha \ \ |\prec_1 \restriction S(\alpha)| \leq |\prec_2 \restriction S(\alpha)|.$$

Proof. Fix $\prec_1$ and $\prec_2$ so that $|\prec_1| \leq |\prec_2|$. Define a function g by $g(s) = t$ where if $s : \omega_1 \to \omega_1$ then $t : \omega_1 \to \omega_1$ and $\forall \alpha \ \ |s(\alpha)|_{\prec_1} = |t(\alpha)|_{\prec_2}$. Consider $s_1, s_2 : \omega_1 \to \omega_1$ such that $s_1 \sim s_2$ and let $t_1 = g(s_1)$ and $t_2 = g(s_2)$. $\forall^*_{W^1_1} \alpha$

$$\begin{aligned} & s_1(\alpha) = s_2(\alpha) \\ \Rightarrow \quad & |s_1(\alpha)|_{\prec_1} = |s_2(\alpha)|_{\prec_1} \\ \Rightarrow \quad & |t_1(\alpha)|_{\prec_2} = |t_2(\alpha)|_{\prec_2} \\ \Rightarrow \quad & t_1(\alpha) = t_2(\alpha) \end{aligned}$$

thus $t_1 \sim t_2$. We are justified, then, in viewing g as a function from ω_2 to ω_2. By Lemma 2.25 the supercompactness measure is closed under g. That is

$$\forall^*_{\mu_2} S \ \ (\eta \in S \Rightarrow g(\eta) \in S).$$

Let $\mathcal{A} \in \mu_2$ be closed under g. Fix S such that $[S] \in \mathcal{A}$. For $\alpha \in \omega_1$ we define a mapping, $\pi_\alpha : (S(\alpha), \prec_1) \to (S(\alpha), \prec_2)$ by $\pi_\alpha(s(\alpha)) = g(s)(\alpha)$. What remains to be shown is that for almost all α π_α is well defined and order preserving. To this end, consider $S, T \in [S]$. We make the following observations:

1. For each pair, $s_i, s_j \in S$, there is a c.u.b. set $C_{i,j}$ so that for any $\alpha, \beta \in C_{i,j}$,

$$s_i(\alpha) < s_j(\alpha) \iff s_i(\beta) < s_j(\beta).$$

2. For each $\eta \in [S]$ there is a corresponding $s \in S$ and $t \in T$ so that $[s] = [t] = \eta$. Thus there is some C_η, a c.u.b. set so that

$$\forall \alpha \in C_\eta \ \ s(\alpha) = t(\alpha).$$

3. For each $\eta \in [S]$, $g(\eta) \in [S]$. Letting $[s] = \eta$, there is some D_η so that

$$\forall \alpha \in D_\eta \ \ g(s)(\alpha) \in S(\alpha).$$

Let $C = (\bigcap_{i,j\in\omega} C_{i,j}) \cap (\bigcap_{\eta\in S} C_\eta) \cap (\bigcap_{\eta\in S} D_\eta)$ and fix $\alpha \in C$. By the construction of C we have that $S(\alpha) = T(\alpha)$. Also, for any $\gamma \in S(\alpha)$ there is precisely one element $s \in S$ so that $s(\alpha) = \gamma$. Finally, for any $\gamma \in S(\alpha)$ $\pi_\alpha(\gamma) \in S(\alpha)$.

Armed with these three observations we now show that π_α is an embedding. If $\gamma_1, \gamma_2 \in S(\alpha)$ with $\gamma_1 \prec_1 \gamma_2$ then $\pi_\alpha(\gamma_1), \pi_\alpha(\gamma_2) \in S(\alpha)$. Moreover, $\pi_\alpha(\gamma_1) \prec_2 \pi_\alpha(\gamma_2)$. Thus π_α is an embedding and the desired result follows. ∎

Lemma 5.6 *Let $\prec_1, \prec_2$ be well-orderings of ω_1 and $f_1, f_2 : \omega_1 \to \omega_1$ such that $\forall^*_{W_1^1}\alpha \ \ |f_1(\alpha)|_{\prec_1} \le |f_2(\alpha)|_{\prec_2}$. Then*

$$\forall^*_{\mu_2} S \ \ \forall^*_{W_1^1}\alpha \ \ |f_1(\alpha)|_{\prec_1 \restriction S(\alpha)} \le |f_2(\alpha)|_{\prec_2 \restriction S(\alpha)}$$

Proof. Let $\prec_1, f_1, \prec_2$ and f_2 be as in the hypothesis. We define a function g by $g(s) = t$ where for each $\alpha \in \omega_1$, if there exists some $\epsilon \in \omega_1$ so that $|s(\alpha)|_{\prec_1} = |\epsilon|_{\prec_2}$ then we let $t(\alpha) = \epsilon$. If no such ϵ exists for a particular α then we let $t(\alpha) = 0$.

Then with a slight modification of the argument from Lemma 5.5, we have that g is well defined on classes of functions so g takes ω_2 to ω_2. Let $\mathcal{A} \in \mu_2$ be both closed under g and concentrate on $[f_1]$ and $[f_2]$. If $s : \omega_1 \to \omega_1$ is such that $\forall^*_{W^1_1}\alpha \;\; s(\alpha) \prec_1 f_1(\alpha)$ then $\forall^*_{W^1_1}\alpha \;\; g(s)(\alpha) \prec_2 f_2(\alpha)$. To see that this is true, fix such an s and let $t = g(s)$. Then

$$\forall^*_{W^1_1}\alpha \;\; |s(\alpha)|_{\prec_1} = |t(\alpha)|_{\prec_2}.$$

If it were the case that $\forall^*_{W^1_1}\alpha \;\; t(\alpha) \succeq_2 f_2(\alpha)$ then

$$\forall^*_{W^1_1}\alpha \;\; |s(\alpha)|_{\prec_1} < |f_1(\alpha)|_{\prec_1} \leq |f_2(\alpha)|_{\prec_2} \leq |t(\alpha)|_{\prec_2},$$

a contradiction. Fix, now, $[S] \in \mathcal{A}$ and let S and T both represent $[S]$. As before, a c.u.b. set C may be constructed so that $\forall \alpha \in C$

1. $T(\alpha) = S(\alpha)$.

2. $\forall \gamma \in S(\alpha) \;\; \exists! s \in S \;\; s(\alpha) = \gamma$.

3. $\forall \gamma \in S(\alpha) \;\; \pi_\alpha(\gamma) \in S(\alpha)$ where $\pi_\alpha(s(\alpha)) = g(S)(\alpha)$.

Then fixing $\alpha \in C$ and letting $\gamma_1, \gamma_2 \in S(\alpha)$ with $\gamma_1 \prec_1 \gamma_2 \prec_1 f_1(\alpha)$, we have that $\pi_\alpha(\gamma_1) \prec_2 \pi_\alpha(\gamma_2) \prec_2 f_2(\alpha)$. Thus π_α is an embedding from

$$\prec_1 \restriction \{\gamma \mid \gamma \in S(\alpha) \text{ and } \gamma \prec_1 f_1(\alpha)\}$$

to

$$\prec_2 \restriction \{\gamma \mid \gamma \in S(\alpha) \text{ and } \gamma \prec_2 f_2(\alpha)\}.$$

So

$$|f_1(\alpha)|_{\prec_1 \restriction S(\alpha)} \leq |f_2(\alpha)|_{\prec_2 \restriction S(\alpha)}.$$

∎

Another result which will be needed is an extension of Kunen's Theorem (Theorem 2.12).

Proposition 5.7 *Let $f : \omega_1^n \to \omega_1$. There exists a $\gamma \in \omega_1$ and well-orderings $\prec_1, \prec_2, \ldots, \prec_n$, of ω_1 so that*

$$\forall^*_{W_1^n}(\alpha_1, \ldots, \alpha_n) \;\; f(\alpha_1, \ldots, \alpha_n) = |\cdots||\gamma|_{\prec_1\restriction\alpha_1}|_{\prec_2\restriction\alpha_2}\cdots|_{\prec_n\restriction\alpha_n}.$$

Proof. Fix $f : \omega_1^n \to \omega_1$. Using the same argument as in Theorem 2.14, we have that there is some well-ordering $\prec_n$ of ω_1 so that

$$\forall^*_{W_1^n}(\alpha_1, \ldots, \alpha_n) \;\; f(\alpha_1, \ldots, \alpha_n) < |\prec_n\restriction \alpha_n|.$$

Thus, there is some function $f' : \omega_1^n \to \omega_1$ and a c.u.b. set $\hat{C}$ so that

$$\forall(\alpha_1, \ldots, \alpha_n) \in \hat{C} \;\; f(\alpha_1, \ldots, \alpha_n) = |f'(\alpha_1, \ldots, \alpha_n)|_{\prec_n\restriction\alpha_n}$$

Fix $(\alpha_1, \ldots, \alpha_{n-1}) \in \hat{C}^{n-1}$. Then $\forall\beta \in \hat{C} \;\; f'(\alpha_1, \ldots, \alpha_{n-1}, \beta) < \beta$. The normality of W_1^1 yields an ordinal $\gamma_{\alpha_1,\ldots,\alpha_{n-1}}$ so that

$$\forall^*_{W_1^1}\beta \;\; f(\alpha_1, \ldots, \alpha_{n-1}, \beta) = |\gamma_{\alpha_1,\ldots,\alpha_{n-1}}|_{\prec_n\restriction\beta}.$$

Such a γ exists for any $(n-1)$-tuple of ordinals from $\hat{C}$ and so we may define $f_{n-1} : \omega_1^{n-1} \to \omega_1$ by $f_{n-1}(\alpha_1, \ldots, \alpha_{n-1}) = \gamma_{\alpha_1,\ldots,\alpha_{n-1}}$ if $(\alpha_1, \ldots, \alpha_{n-1}) \in \hat{C}^{n-1}$ and define f_{n-1} to be 0 otherwise. Then

$$\forall^*_{W_1^n}(\alpha_1, \ldots, \alpha_{n-1}, \alpha_n) \;\; f(\alpha_1, \ldots, \alpha_{n-1}, \alpha_n) = |f_{n-1}(\alpha_1, \ldots, \alpha_{n-1})|_{\prec_n\restriction\alpha_n}.$$

We now do precisely the same thing with the function f_{n-1}, producing a function $f_{n-2} : \omega_1^{n-2} \to \omega_1$ and a well-ordering $\prec_{n-1}$, so that

$$\forall^*_{W_1^{n-1}}(\alpha_1, \dots, \alpha_{n-1}) \;\; f_{n-1}(\alpha_1, \dots, \alpha_{n-1}) = |f_{n-2}(\alpha_1, \dots, \alpha_{n-2})|_{\prec_{n-1} \restriction \alpha_{n-1}}.$$

So we have $\forall^*_{W_1^n}(\alpha_1, \dots, \alpha_n)$

$$f(\alpha_1, \dots, \alpha_n) = ||f_{n-2}(\alpha_1, \dots, \alpha_{n-2})|_{\prec_{n-1} \restriction \alpha_{n-1}}|_{\prec_n \restriction \alpha_n}.$$

Continuing we will eventually have a function $f_1 : \omega_1 \to \omega_1$, to consider. By Kunen's Theorem (Theorem 2.12) and the comments following Kunen's Theorem there is a well-ordering $\prec_1$ and a $\gamma \in \omega_1$ so that

$$\forall^*_{W_1^1}\alpha_1 \;\; f_1(\alpha_1) = |\gamma|_{\prec_1 \restriction \alpha_1}$$

and so

$$\forall^*_{W_1^n}(\alpha_1, \dots, \alpha_n) \;\; f(\alpha_1, \dots, \alpha_n) = |\cdots||\gamma|_{\prec_1 \restriction \alpha_1}|_{\prec_2 \restriction \alpha_2} \cdots|_{\prec_n \restriction \alpha_n}.$$

∎

5.2 The Lower Bound for $j_{\mu_2}(\omega_n)$ Revisited

Lemma 5.8

$$j_{\mu_2}(\omega_1) \geq \omega_3$$

Proof. Let $\theta \in \omega_3$ and let $f : \omega_1^2 \to \omega_1$ represent θ. By Proposition 5.7 there is some $\gamma \in \omega_1$ and well-orderings $\prec_1, \prec_2$ of ω_1 so that

$$\forall^*_{W_1^2}(\alpha, \beta) \;\; f(\alpha, \beta) = ||\gamma|_{\prec_1 \restriction \alpha}|_{\prec_2 \restriction \beta}.$$

Fix $[S] \in P_{\omega_1}(\omega_2)$. The first claim is that for S representing $[S]$ it is the case that there is some $\lambda^f_S \in \omega_1$ so that

$$\forall^*_{W^1_1}\alpha \ \ ||\gamma|_{\prec_1\restriction\alpha}|_{\prec_2\restriction S(\alpha)} = \lambda^f_S.$$

Indeed, let $g : \omega_1 \to \omega_1$ be defined by $g(\alpha) = |\gamma|_{\prec_1\restriction\alpha}$. Then for any $s \in S$ there is a c.u.b. set C_s so that

$$\forall\alpha \in C_s \ \ s(\alpha) \preceq_2 g(\alpha) \ \text{ or } \ \forall\alpha \in C_s \ \ g(\alpha) \prec_2 s(\alpha).$$

Letting $C = \bigcap_{s\in S} C_s$ we have that

$$\forall\alpha,\beta \in C \ \ \forall s \in S \ \ s(\alpha) \preceq_2 g(\alpha) \iff s(\beta) \preceq_2 g(\beta)$$

so $\forall\alpha,\beta \in C$

$$|\prec_2\restriction \{s(\alpha) \in S(\alpha) \ : \ s(\alpha) \preceq_2 g(\alpha)\}| = |\prec_2\restriction \{s(\beta) \in S(\beta) \ : \ s(\beta) \preceq_2 g(\beta)\}|.$$

Thus

$$\forall\alpha,\beta \in C \ \ ||\gamma|_{\prec_1\restriction\alpha}|_{\prec_2\restriction S(\alpha)} = ||\gamma|_{\prec_1\restriction\beta}|_{\prec_2\restriction S(\beta)} = \lambda^f_S$$

The second claim is that if both S and T represent a set $[S]$ then $\lambda^f_S = \lambda^f_T$. Since $S \sim T$ we have that $\exists C$

$$\begin{array}{rc} & \forall s \in S \ \ \forall t \in T \ \ (s \sim t \iff \forall\alpha \in C \ \ s(\alpha) = t(\alpha)) \\ \Rightarrow & \forall\alpha \in C \ \ ||\gamma|_{\prec_1\restriction\alpha}|_{\prec_2\restriction S(\alpha)} = ||\gamma|_{\prec_1\restriction\alpha}|_{\prec_2\restriction T(\alpha)} \\ \Rightarrow & \lambda^f_S = \lambda^f_T \end{array}$$

Thus, λ^f_S is well defined. What must now be shown is that for two functions $f, h : \omega_2 \to \omega_2$, such that $f \sim h$, $\forall^*_{\mu_2} S \ \ \lambda^f_S = \lambda^h_S$. Let $f, h \in [f]$ with $\gamma^f, \prec^f_1$ and $\prec^f_2$

corresponding to f and $\gamma^h, \prec_1^h$ and $\prec_2^h$ corresponding to h. Then

$$\forall^*_{W_1^2}(\alpha,\beta) \;\; f(\alpha,\beta) = h(\alpha,\beta)$$

$$\Rightarrow \;\; \forall^*_{W_1^2}(\alpha,\beta) \;\; ||\gamma^f|_{\prec_1^f\restriction\alpha}|_{\prec_2^f\restriction\beta} = ||\gamma^h|_{\prec_1^h\restriction\alpha}|_{\prec_2^h\restriction\beta}.$$

By Lemma 5.4

$$\forall^*_{W_1^1}\alpha \;\; ||\gamma^f|_{\prec_1^f\restriction\alpha}|_{\prec_2^f} = ||\gamma^h|_{\prec_1^h\restriction\alpha}|_{\prec_2^h}$$

and by Lemma 5.6

$$\forall^*_{\mu_2}S \;\; \forall^*_{W_1^1}\alpha \;\; ||\gamma^f|_{\prec_1^f\restriction\alpha}|_{\prec_2^f\restriction S(\alpha)} = ||\gamma^h|_{\prec_1^h\restriction\alpha}|_{\prec_2^h\restriction S(\alpha)}.$$

Thus $\forall^*_{\mu_2}S \;\; \lambda_S^f = \lambda_S^h$. So for almost all sets S we may associate the ordinal λ_S to S.

We define the map $\pi : \omega_3 \to j_{\mu_2}(\omega_2)$ by $\pi([f]) = F$ where $F(S) = \lambda_S$. By the above comments, this mapping is well defined and a similar argument demonstrates that this mapping is order preserving as well. ∎

Lemma 5.9

$$j_{\mu_2}(\omega_n) \geq \omega_{2n+1}$$

Proof. We will show that $j_{\mu_2}(\omega_3) \geq \omega_7$. This case is general enough to demonstrate the procedure involved in proving the result for an arbitrary ω_n.

Fix $\theta < \omega_7$ and let $f : \omega_1^6 \to \omega_1$ represent θ. Then there are well-orderings $\prec_i$, $1 \leq i \leq 6$, and an ordinal $\gamma \in \omega_1$ so that

$$\forall^*_{W_1^6}\vec{\alpha} \;\; f(\vec{\alpha}) = |\cdots|\gamma|_{\prec_1\restriction\alpha_1}\cdots|_{\prec_6\restriction\alpha_6}.$$

To the class of functions $[f]$ we associate the class $[F]$ so that

$$\forall^*_{S_1^1}\overline{s} \;\; \forall^*_{W_1^1}\delta \;\; \forall^*_{W_1^2}(\alpha,\beta) \;\; F(S_{\overline{s},\delta})(\alpha,\beta) = \lambda_{\overline{s},\delta,\alpha,\beta}$$

where $\;\forall^*_{W_1^1}\xi \;\; ||||||\gamma|_{\prec_1\restriction\delta}|_{\prec_2\restriction\overline{s}(\delta)}|_{\prec_3\restriction\alpha}|_{\prec_4\restriction\overline{s}(\alpha)}|_{\prec_5\restriction\xi}|_{\prec_6\restriction S_{\overline{s},\beta}(\xi)} = \lambda_{\overline{s},\delta,\alpha,\beta}.$

It is not difficult to see that for a fixed $\overline{s}$ with $\rho : \omega_1 \to [\overline{s}]$, and δ, α and β with $S_{\overline{s},\beta}$ representing $[S_{\overline{s},\beta}]$, there is such a stabilized $\lambda_{\overline{s},\delta,\alpha,\beta}$. Indeed, defining $k : \omega_1 \to \omega_1$ by

$$k(\xi) = |||||\gamma|_{\prec_1 \restriction \delta}|_{\prec_2 \restriction \overline{s}(\delta)}|_{\prec_3 \restriction \alpha}|_{\prec_4 \restriction \overline{s}(\alpha)}|_{\prec_5 \restriction \xi}$$

we have, by an argument similar to the one in the previous lemma, that

$$\exists \lambda_{\overline{s},\delta,\alpha,\beta} \;\; \forall^*_{W_1^1} \xi \;\; |k(\xi)|_{\prec_6 \restriction S_{\overline{s},\beta}(\xi)} = \lambda_{\overline{s},\delta,\alpha,\beta}.$$

By previous arguments (Lemma 3.2, Theorem 4.2, Lemma 5.8) we are justified in suppressing mention of the bijection ρ and the representatives, $\overline{s}$ and $S_{\overline{s},\beta}$ in the definition of $\lambda_{\overline{s},\delta,\alpha,\beta}$. What must be demonstrated is that if both $f, h : \omega_1^6 \to \omega_1$, are functions such that $f \sim h$ and F corresponds to f and H corresponds to h, then

$$\forall^*_{S_1^1} \overline{s} \;\; \forall^*_{W_1^1} \delta \;\; \forall^*_{W_1^2} (\alpha, \beta) \;\; F(S_{\overline{s},\delta})(\alpha, \beta) = H(S_{\overline{s},\delta})(\alpha, \beta).$$

To this end, for $1 \leq i \leq 6$, let $\prec_i^f$ be associated to f and $\prec_i^h$ be associated to h with respective ordinals γ^f and γ^h. Then

$$\forall^*_{W_1^6} \vec{\alpha} \;\; |\cdots|\gamma^f|_{\prec_1^f \restriction \alpha_1} \cdots|_{\prec_6^f \restriction \alpha_6} \quad = \quad |\cdots|\gamma^h|_{\prec_1^h \restriction \alpha_1} \cdots|_{\prec_6^h \restriction \alpha_6}$$

By Lemma 5.4

$$\forall^*_{W_1^5} (\alpha_1, \ldots, \alpha_5) \;\; |\cdots|\gamma^f|_{\prec_1^f \restriction \alpha_1} \cdots|_{\prec_6^f} \quad = \quad |\cdots|\gamma^h|_{\prec_1^h \restriction \alpha_1} \cdots|_{\prec_6^h}$$

and so $\exists D_1 \in S_1^1 \;\; \forall \overline{s} \in D_1 \;\; \forall^*_{W_1^3} (\delta, \alpha, \xi)$

$$\begin{aligned} & ||||||\gamma^f|_{\prec_1^f \restriction \delta}|_{\prec_2^f \restriction \overline{s}(\delta)}|_{\prec_3^f \restriction \alpha}|_{\prec_4^f \restriction \overline{s}(\alpha)}|_{\prec_5^f \restriction \xi}|_{\prec_6^f} \\ = \;\; & ||||||\gamma^h|_{\prec_1^h \restriction \delta}|_{\prec_2^h \restriction \overline{s}(\delta)}|_{\prec_3^h \restriction \alpha}|_{\prec_4^h \restriction \overline{s}(\alpha)}|_{\prec_5^h \restriction \xi}|_{\prec_6^h}. \end{aligned}$$

At this point we need a slight modification of Lemma 5.6. Assume, without loss of generality, that $| \prec_6^f | \leq | \prec_6^h |$. We define g_1 by $g_1(s) = t$ where $s, t : \omega_1 \to \omega_1$ and $\forall \xi \in \omega_1 \ \ |s(\xi)|_{\prec_6^f} = |t(\xi)|_{\prec_6^h}$. We also define g_2 by $g_2(s) = t$ where for each $\xi \in \omega_1$, if there exists some $\epsilon \in \omega_1$ so that $|s(\xi)|_{\prec_1} = |\epsilon|_{\prec_2}$ then we let $t(\xi) = \epsilon$. If no such ϵ exists for a particular ξ then we let $t(\xi) = 0$.

By an argument similar to that given in Lemma 5.5, both g_1 and g_2 are well defined for classes of functions and may be viewed as functions from ω_2 to ω_2. Then there is some $D_2 \in S_1^1$ which is closed under g_1, g_2 in the sense that if $\overline{s} \in D_2$ then

$$\forall^*_{W_1^1} \beta \ \ s \in S_{\overline{s},\beta} \Rightarrow g_1(s), g_2(s) \in S_{\overline{s},\beta}.$$

This is obvious since $\exists \mathcal{A} \in \mu_2$ such that $\mathcal{A}$ is closed under g_1 and g_2, and there is some $D_2 \in S_1^1$ so that

$$\forall \overline{s} \in D_2 \ \ \forall^*_{W_1^1} \beta \ \ S_{\overline{s},\beta} \in \mathcal{A}.$$

Fix $\overline{s} \in D_1 \cap D_2$ and δ and α so that $\forall^*_{W_1^1} \xi$

$$\begin{aligned} & ||||||\gamma^f|_{\prec_1^f \restriction \delta}|_{\prec_2^f \restriction \overline{s}(\delta)}|_{\prec_3^f \restriction \alpha}|_{\prec_4^f \restriction \overline{s}(\alpha)}|_{\prec_5^f \restriction \xi}|_{\prec_6^f} \\ = \ & ||||||\gamma^h|_{\prec_1^h \restriction \delta}|_{\prec_2^h \restriction \overline{s}(\delta)}|_{\prec_3^h \restriction \alpha}|_{\prec_4^h \restriction \overline{s}(\alpha)}|_{\prec_5^h \restriction \xi}|_{\prec_6^h}. \end{aligned}$$

Define $k^f, k^h : \omega_1 \to \omega_1$ by

$$\begin{aligned} k^f(\xi) &= |||||\gamma^f|_{\prec_1^f \restriction \delta}|_{\prec_2^f \restriction \overline{s}(\delta)}|_{\prec_3^f \restriction \alpha}|_{\prec_4^f \restriction \overline{s}(\alpha)}|_{\prec_5^f \restriction \xi} \\ \text{and} \quad k^h(\xi) &= |||||\gamma^h|_{\prec_1^h \restriction \delta}|_{\prec_2^h \restriction \overline{s}(\delta)}|_{\prec_3^h \restriction \alpha}|_{\prec_4^h \restriction \overline{s}(\alpha)}|_{\prec_5^h \restriction \xi} \end{aligned}$$

Then

$$\forall^*_{W_1^1} \xi \ \ |k^f(\xi)|_{\prec_6^f} = |k^h(\xi)|_{\prec_6^h}.$$

In particular, $\forall^*_{W^1_1}\xi \;\; |k^f(\xi)|_{\prec^f_6} \leq |k^h(\xi)|_{\prec^h_6}$. Since $\overline{s} \in D_2$ it is the case that $\forall^*_{W^1_1}\beta$ if $s \in S_{\overline{s},\beta}$ then $g_1(s) \in S_{\overline{s},\beta}$. Fix such a β. By Lemma 5.6 we have that a c.u.b. set C exists so that $\forall \xi \in C$ one may construct an embedding π_ξ witnessing that $|k^f(\xi)|_{\prec^f_6 \restriction S_{\overline{s},\beta}(\xi)} \leq |k^h(\xi)|_{\prec^h_6 \restriction S_{\overline{s},\beta}(\xi)}$. So

$$\forall^*_{W^1_1}\beta \;\; \forall^*_{W^1_1}\xi \;\; |k^f(\xi)|_{\prec^f_6 \restriction S_{\overline{s},\beta}(\xi)} \leq |k^h(\xi)|_{\prec^h_6 \restriction S_{\overline{s},\beta}(\xi)}.$$

Similarly, using the fact that $\forall^*_{W^1_1}\xi \;\; |k^f(\xi)|_{\prec^h_6} \leq |k^h(\xi)|_{\prec^f_6}$ and the fact that D_2 is closed under g_2, we have that

$$\forall^*_{W^1_1}\beta \;\; \forall^*_{W^1_1}\xi \;\; |k^f(\xi)|_{\prec^h_6 \restriction S_{\overline{s},\beta}(\xi)} \leq |k^h(\xi)|_{\prec^f_6 \restriction S_{\overline{s},\beta}(\xi)}.$$

Thus

$$\forall^*_{S^1_1}\overline{s} \;\; \forall^*_{W^1_1}\delta \;\; \forall^*_{W^2_1}(\alpha,\beta) \;\; F(S_{\overline{s},\delta})(\alpha,\beta) = H(S_{\overline{s},\delta})(\alpha,\beta).$$

The function taking $[f]$ to $[F]$ is well defined and similarly order preserving so $j_{\mu_2}(\omega_3) \geq \omega_7$. This result may be extended in the obvious way. ∎

The key idea in this proof was that using the alternate definition of μ_2 one was able to describe a set S as an ordinal and describe the last element in the finite sequence of ordinals as an element of $P_{\omega_1}(\omega_2)$. The point is that this technique relied on Woodin's Theorem for the equivalence of the two definitions of μ_2. Because of this, it is unclear as to whether this technique may be extended to μ_n for $n > 2$.

CHAPTER 6

UPPER BOUND RESULTS

6.1 The Upper Bound for $P_{\omega_1}(\omega_2)$

It will be shown, in this section, that the results for the measure μ_2 on $P_{\omega_1}(\omega_2)$ are, in fact, sharp. The main result is, as expected, that $j_{\mu_2}(\omega_n) \leq \omega_{2n+1}$. The proof of this result will be completely different than that used in obtaining the lower bound. In fact, this proof will rely on an analysis of iterated ultrapowers by ordinal measures.

Lemma 6.1 *Let δ be a cardinal. Then*

$$j_{\mu_2}(\delta) \leq j_{S_1^1}(j_{W_1^1}(\delta)).$$

Proof. Fix $F : P_{\omega_1}(\omega_2) \to \delta$. The idea is to construct a function $f : \omega_2 \to j_{W_1^1}(\delta)$ corresponding to F and, of course, show that the mapping π, defined by $\pi([F]) = [f]$ is an embedding.

Fix $\eta < \omega_2$ and let $\rho : \omega_1 \to \eta$ be a bijection. Then for any $\alpha < \omega_1$ define $S_\alpha = \rho''(\alpha)$. As always, the choice of ρ may be suppressed. Then

$$\forall^*_{W_1^1}\alpha \quad S_\alpha \in P_{\omega_1}(\omega_2) \text{ and } F(S_\alpha) = \gamma_\alpha < \delta.$$

Define $f(\eta) = [h]$ where $h(\alpha) = F(S_\alpha)$.

Suppose that $F \sim G$ where $F, G : P_{\omega_1}(\omega_2) \to \delta$ and suppose that f corresponds to F and g corresponds to G.

Since $F \sim G$ then there is some D, an ω-cofinal closed unbounded subset of ω_2, so that $\forall \eta \in D$

$$\begin{aligned} & \forall^*_{W_1^1}\alpha \quad F(S_\alpha) = G(S_\alpha) \\ \iff & \forall^*_{W_1^1}\alpha \quad h_F(\alpha) = h_G(\alpha) \\ \iff & [h_F] = [h_G] \\ \iff & f(\eta) = g(\eta) \end{aligned}$$

Here, h_F, h_G are defined in the obvious way. Therefore, the function $\pi : j_{\mu_2}(\omega_2) \to j_{S_1^1}(j_{W_1^1}(\delta))$ defined by $\pi([F]) = [f]$, is well defined. Replacing the assumption that $F \sim G$ with $F < G$ results in the similar proof that π is order preserving. ∎

To get the required upper bound result using this method, it must be demonstrated that $j_{S_1^1}(j_{W_1^1}(\omega_n))$ is suitably bounded.

Lemma 6.2

$$j_{S_1^1}(j_{W_1^1}(\omega_n)) = \omega_{2n+1}$$

Proof. It will first be shown that $j_{W_1^1}(\omega_n) = \omega_{n+1}$. To demonstrate that $j_{W_1^1}(\omega_n) \geq \omega_{n+1}$, for any function $f : \omega_1^n \to \omega_1$ we define the associated function $F : \omega_1 \to \omega_n$ by $F(\alpha_0) = h_{\alpha_0}$ where $h_{\alpha_0} : \omega_1^{n-1} \to \omega_1$ is defined by

$$h_{\alpha_0}(\alpha_1, \ldots, \alpha_{n-1}) = f(\alpha_0, \alpha_1, \ldots, \alpha_{n-1}).$$

It must, of course, be shown that the mapping $f \mapsto F$ is well defined, that is if $f \sim g$ with F associated to f and G associated to g, then $\forall^*_{W_1^1}\alpha_0 \quad F(\alpha_0) \sim G(\alpha_0)$. Let C be a c.u.b. set so that

$$\forall \vec{\alpha} \in C^n \quad f(\vec{\alpha}) = g(\vec{\alpha}).$$

Then $\forall \alpha_0 \in C \ \ \forall (\alpha_1, \ldots, \alpha_{n-1}) \in C^{n-1}$

$$\begin{aligned} & F(\alpha_0)(\alpha_1, \ldots, \alpha_{n-1}) \\ = \quad & f(\alpha_0, \alpha_1, \ldots, \alpha_{n-1}) \\ = \quad & g(\alpha_0, \alpha_1, \ldots, \alpha_{n-1}) \\ = \quad & G(\alpha_0)(\alpha_1, \ldots, \alpha_{n-1}) \end{aligned}$$

We define the mapping, $\pi : \omega_{n+1} \to j_{\omega_1}(\omega_n)$, by $\pi([f]) = [F]$ and observe that, by the above comments, π is well defined. By a similar argument π is order preserving. Thus $j_{W_1^1}(\omega_n) \geq \omega_{n+1}$.

We now show that $j_{W_1^1}(\omega_n) \leq \omega_{n+1}$. We will use induction on the index, n. For the case where $n = 1$, we have, by Theorem 2.14, that the result holds. Let $F : \omega_1 \to \omega_n$ and observe that since $\forall n > 1 \ \ cof(\omega_n) = \omega_2$, the range of the function F is bounded in ω_n. That is, there is some $\eta < \omega_n$ so that $\forall \alpha \in \omega_1 \ \ F(\alpha) < \eta$. Then $[F] < j_{W_1^1}(\eta)$ so

$$j_{W_1^1}(\omega_n) = \sup_{\eta \in \omega_n} j_{W_1^1}(\eta) \leq j_{W_1^1}(\omega_{n-1})^+ = \omega_{n+1}.$$

We now show that $j_{S_1^1}(\omega_n) = \omega_{2n-1}$. In this case we will present a specific example, namely that $j_{S_1^1}(\omega_3) = \omega_5$ and argue that the general case is similar.

Let $F : \omega_2 \to \omega_3$. Then for any class of functions $[g]$ where $g : \omega_1 \to \omega_1$, $[F([g])]$ may be viewed as a class of functions where $F([g]) : \omega_1^2 \to \omega_1$. Fix a partition P on pairs of functions $g, h : \omega_1 \to \omega_1$ of c.t. by

$$P(g, h) = 1 \iff \forall^*_{W_1^2}(\alpha, \beta) \ \ F([g])(\alpha, \beta) < h(\beta)$$

where $F([g]) \in [F([g])]$. Suppose that both $k_1, k_2 \in [F([g])]$. Then $\forall^*_{W_1^2}(\alpha, \beta)$

$$k_1(\alpha, \beta) < g(\beta) \iff k_2(\alpha, \beta) < g(\beta)$$

and so P is well defined. Let C be homogeneous for P and $g : \omega_1 \to C$ of correct type. Then by our observation concerning the fact that P is well defined, we may fix a representative function $F([g])$. One may easily fix a function $h : \omega_1 \to C$ of c.t. so that $[h] > [g]$ and $\forall^*_{W_1^2}(\alpha, \beta) \;\; F([g])(\alpha, \beta) < h(\beta)$. We may assume that g and h weave. Then the pair (g, h) demonstrate that C is homogeneous for the 1-side.

Let $\hat{C} \subset C$ be such that there are ω^2 many elements of C between each point in $\hat{C}$. For $g : \omega_1 \to \hat{C}$ of c.t. define $h_g(\beta) = N^\omega_C(g(\beta))$ where $N^\omega_C(\xi)$ is the ωth element of C after ξ. Then $h_g : \omega_1 \to C$ is of c.t. and $[h_g] > [g]$. Thus

$$\forall g : \omega_1 \to \hat{C} \text{ of c.t. } \quad \forall^*_{W_1^2}(\alpha, \beta) \;\; F([g])(\alpha, \beta) < h_g(\beta)$$

$$\Rightarrow \quad \forall g : \omega_1 \to \hat{C} \text{ of c.t. }, \forall^*_{W_1^2}(\alpha, \beta) \;\; F([g])(\alpha, \beta) < ((\alpha, \beta) \to N^\omega_C(g(\beta)))$$

$$\Rightarrow \qquad [F] < [[g] \to [(\alpha, \beta) \to N^\omega_C(g(\beta))]]$$

$$\Rightarrow \qquad \exists \prec_1, \text{ a well-ordering of } \omega_1 \text{ such that}$$

$$[F] < [[g] \to [(\alpha, \beta) \to | \prec_1 \restriction g(\beta)|]]$$

$$\Rightarrow \qquad [F] < [[g] \to [(\alpha, \beta) \to g(\beta)]]^+$$

Let $[K_1] < [[g] \to [(\alpha, \beta) \to g(\beta)]]$. Define a partition on pairs of functions, $h, g : \omega_1 \to \omega_1$ of c.t. by

$$P(h, g) = 1 \iff \forall^*_{W_1^2}(\alpha, \beta) \;\; K_1([g])(\alpha, \beta) < h(\beta)$$

and let C be homogeneous for P. Fix $g : \omega_1 \to C$ of correct type. Let $K_1([g]) \in [K_1([g])]$ and notice that

$$\forall^*_{W_1^2}(\alpha, \beta) \;\; K_1([g])(\alpha, \beta) < g(\beta).$$

Let $\hat{C} \subset C$ be such that there are ω^2 many elements of C between each point in $\hat{C}$. Then $K_1([g])$ and g may be assumed to weave on $\hat{C}$. Define $h : \omega_1 \to C$ by $h(\beta) = N_C^\omega(K_1([g])(1)(\beta))$. Then h is of c.t. and $[h] < [g]$ and so (h, g) witnesses that C is homogeneous for the 1-side.

Thus

$$\forall g : \omega_1 \to \hat{C} \text{ of c.t.} \quad \forall^*_{W_1^2}(\alpha, \beta) \;\; K_1([g])(\alpha, \beta) < N_C^\omega(\beta).$$

By the usual arguments,

$$[K_1] < [[g] \to [(\alpha, \beta) \to \beta]]^+.$$

Letting $[K_2] < [(\alpha, \beta) \to \beta]]$ we define a partition on functions $g, h : \omega_1 \to \omega_1$ of c.t. by

$$P(g, h) = 1 \iff \forall^*_{W_1^2}(\alpha, \beta) \;\; K_2([g])(\alpha, \beta) < h(\alpha).$$

The standard argument yields that a homogeneous set C is homogeneous for the 1-side. Then, as before,

$$[K_2] < [[g] \to [(\alpha, \beta) \to g(\alpha)]]^+.$$

Continuing we are able to produce the following table:

$$\begin{aligned}
\omega_5 &\geq j_{S_1^1}(\omega_3) \\
\omega_4 &\geq [[g] \to [(\alpha, \beta) \to g(\beta)]] \\
\omega_3 &\geq [[g] \to [(\alpha, \beta) \to \beta]] \\
\omega_2 &\geq [[g] \to [(\alpha, \beta) \to g(\alpha)]] \\
\omega_1 &\geq [[g] \to [(\alpha, \beta) \to \alpha]]
\end{aligned}$$

We now proceed to get the lower bounds for each level. Fix $\xi < \omega_1$ and observe that $\forall^*_{W_1^2}(\alpha, \beta) \;\; \alpha > \xi$ so

$$\begin{aligned} & [[g] \to [(\alpha,\beta) \to \alpha]] > [[g] \to [(\alpha,\beta) \to \xi]] \\ \Rightarrow \qquad & [[g] \to [(\alpha,\beta) \to \alpha]] \geq \omega_1. \end{aligned}$$

Fix $\eta < \omega_2$ and represent η by a function $f : \omega_1 \to \omega_1$. If $f \sim f'$ then $\forall^*_{W_1^2}(\alpha, \beta) \;\; f(\beta) = f'(\beta)$ so

$$[[g] \to [(\alpha,\beta) \to f(\beta)]] = [[g] \to [(\alpha,\beta) \to f'(\beta)]]$$

Notice, also, that $\forall^*_{S_1^1}[g] \;\; [g] > [f]$ so $\forall^*_{W_1^2}(\alpha, \beta) \;\; g(\alpha) > f(\alpha)$ and thus

$$\begin{aligned} & [[g] \to [(\alpha,\beta) \to g(\alpha)]] > [[g] \to [(\alpha,\beta) \to f(\alpha)]] \\ \Rightarrow \qquad & [[g] \to [(\alpha,\beta) \to g(\alpha)]] \geq \omega_2 \end{aligned}$$

Fix $\lambda \in \omega_3$ and let $l : \omega_1^2 \to \omega_1$ represent λ. If $l \sim l'$ then there is some c.u.b. set C so that

$$\forall(\alpha,\beta) \in C^2 \;\; l(\alpha,\beta) = l'(\alpha,\beta).$$

Then for any $g : \omega_1 \to C$ of c.t. we have that

$$\forall^*_{W_1^2}(\alpha,\beta) \in C^2 \;\; l(\alpha, g(\alpha)) = l'(\alpha, g(\alpha))$$

so $[[g] \to [(\alpha,\beta) \to l(\alpha, g(\alpha))]]$ is well defined. Also, $\forall^*_{W_1^2}(\alpha, \beta) \;\; \beta > l(\alpha, g(\alpha))$ so

$$\begin{aligned} & [[g] \to [(\alpha,\beta) \to \beta]] > [[g] \to [(\alpha,\beta) \to l(\alpha, g(\alpha))]] \\ \Rightarrow \qquad & [[g] \to [(\alpha,\beta) \to \beta]] \geq \omega_3. \end{aligned}$$

Let $m : \omega_1^3 \to \omega_1$ represent an ordinal $\delta < \omega_4$. The class

$$[[g] \to [(\alpha, \beta) \to m(\alpha, g(\alpha), \beta)]]$$

is well defined and

$$\begin{aligned} & \forall^*_{S^1_1}[g] \ \ \forall^*_{W^2_1}(\alpha, \beta) \ \ g(\beta) > m(1)(\beta) \\ \Rightarrow \ \ & \forall^*_{S^1_1}[g] \ \ \forall^*_{W^2_1}(\alpha, \beta) \ \ g(\beta) > m(\alpha, g(\alpha), \beta) \end{aligned}$$

So

$$\begin{aligned} & [[g] \to [(\alpha, \beta) \to g(\beta)]] > [[g] \to [(\alpha, \beta) \to m(\alpha, g(\alpha), \beta)]] \\ \Rightarrow \qquad & [[g] \to [(\alpha, \beta) \to g(\beta)]] \geq \omega_4. \end{aligned}$$

Finally, For $\gamma \in \omega_5$ with $r : \omega_1^4 \to \omega_1$ representing γ, we have that

$$[[g] \to [(\alpha, \beta) \to r(\alpha, g(\alpha), \beta, g(\beta))]]$$

is well defined. Using the fact that $j_{W^2_1}(\omega_1) = \omega_3$ (see the proof of Theorem 2.14) we have that

$$\begin{aligned} & [[g] \to [(\alpha, \beta) \to \omega_1]] > [[g] \to [(\alpha, \beta) \to r(\alpha, g(\alpha), \beta, g(\beta))]] \\ \Rightarrow \qquad & j_{S^1_1}(\omega_3) > [[g] \to [(\alpha, \beta) \to r(\alpha, g(\alpha), \beta, g(\beta))]] \\ \Rightarrow \qquad & j_{S^1_1}(\omega_3) \geq \omega_5 \end{aligned}$$

We will now glue parts 1 and 2 together to get the desired result. Indeed

$$\begin{aligned} & j_{S^1_1}(j_{W^1_1}(\omega_n)) \\ = \ \ & j_{S^1_1}(\omega_{n+1}) \\ = \ \ & \omega_{2(n+1)-1} \\ = \ \ & \omega_{2n+1} \end{aligned}$$

∎

Corollary 6.3

$$j_{\mu_2}(\omega_n) = \omega_{2n+1}$$

Proof. By Theorem 4.2 we have that $j_{\mu_2}(\omega_n) \geq \omega_{2n+1}$. Also, by Lemma 6.1 and the preceding lemma,

$$j_{\mu_2}(\omega_n) \leq j_{S_1^1}(j_{W_1^1}(\omega_n)) \leq \omega_{2n+1}.$$

∎

6.2 Generic Codes and Suslin Cardinals

In this section, the reader will be introduced to a powerful idea related to supercompactness measures, the notion of Generic Codes. They idea is due to Kechris and Woodin [KW]. For the benefit of the reader, a detailed look at the first chapter will be presented.

Definition 6.4 *An ordinal λ is said to be* reliable *if there is a scale $\{\varphi_i\}_{i\in\omega}$ on a set W so that the following conditions are satisfied:*

1. *φ_0 is a norm onto λ.*

2. *$\forall i \in \omega \;\; \varphi_i : W \to \lambda$*

3. The relations

$$x, y \in W \text{ and } \varphi_0(x) < \varphi_0(y)$$

$$x, y \in W \text{ and } \varphi_0(x) \leq \varphi_0(y)$$

admit scales. We say, then, that the pair $(\{\varphi_i\}, W)$ *witnesses the reliability of* λ.

Definition 6.5 *We call* $S \in P_{\omega_1}(\lambda)$ ξ-honest *where* $\xi \in S$*, if there is some* $x \in W$ *so that* $\varphi_0(x) = \xi$ *and* $\forall i \in \omega \;\; \varphi_i(x) \in S$.

A set $S \in P_{\omega_1}(\lambda)$ *is* honest *if* S *is* ξ*-honest for all* $\xi \in S$.

A simple observation is that any element of ω^ω actually codes a countable sequence of reals. This coding may be accomplished in various ways. For example, consider $x \in \omega^\omega$, $x = (x_0, x_1, x_2, \dots)$. To find $(x)_n$ (the nth real which x codes) we take p to be the $(n+2)$nd prime and say that $(x)_n = (x_p, x_{2p}, x_{3p}, \dots)$. It matters not which coding is adopted but for the record, we will take the above coding.

Theorem 6.6 (Kechris-Woodin) *Let* λ *be reliable with witness* $(\{\varphi_i\}, W)$*. Then there exist l.s. (Lipschitz) continuous functions* $G_0, G : \lambda^\omega \to \omega^\omega$ *such that*

1. Range$(G_0) \subset W$ *and* $\forall f \in \lambda^\omega$

$$\{f(0), f(1), \dots\} \text{ is } f(0)\text{-honest } \Rightarrow \varphi_0(G_0(f)) = f(0)$$

2. Range$(G) \subset \{x \mid \forall n \in \omega \;\; (x)_n \in W\}$ *and* $\forall f \in \lambda^\omega$

$$\{f(0), f(1), \dots\} \text{ is honest } \Rightarrow \forall n \in \omega \;\; \varphi_0((G(f))_n) = f(n)$$

Proof. We will prove part 1 and then make some brief comments to explain the differences in the proof of part 2. From the comments preceding Theorem 2.12, we have that there is some tree coming from the scale $\{\varphi_i\}$. Taking $T \subset (\omega^{<\omega}, \lambda^{<\omega})$ to be such a tree, we may define, for $\xi < \lambda$,

$$T_\xi = \{((x(0), \dots, x(n)), (u(0), \dots, u(n))) \mid u(0) = \xi\}.$$

Consider the following game:

$$\begin{array}{c|ccccccc} \mathrm{I} & f(0) & & f(1) & & f(2) & \cdots \\ \mathrm{II} & & w_0, h(0) & & w_1, h(1) & & \cdots \end{array}$$

where $f(i), h(i) \in \lambda$ and $w_i \in \omega$. We say that player II wins if and only if

1. $(w, h) \in [T_{f(0)}]$ and

2. $\forall v \in \omega^\omega \quad \big(v \in p[T_{f(0)} \restriction \{f(0), f(1), \dots\}] \Rightarrow \varphi_0(v) \le \varphi_0(w)\big)$

The first claim is that this game is determined. To prove this we must show that the relation $R \subset \lambda^\omega \times \omega^\omega \times \lambda^\omega$, defined by

$$\begin{aligned} (f, w, h) \in R \quad \Rightarrow \quad & \text{The triple, } (f, w, h)\text{, occurs as a run of the game} \\ & \text{with } (f, w, h) \text{ a win for player II} \end{aligned}$$

and its negation, admit scales. Then, appealing to a theorem of Moschovakis [M1], the game is determined. Dissecting R we have that $(f, w, h) \in R \iff$

$$(w, h) \in [T_{f(0)}] \text{ and } \big\{\forall v \left[v \notin [T_{f(0)} \restriction \{f(0), f(1), \dots\}]\right] \text{ or } \varphi_0(v) \le \varphi_0(w)\big\}.$$

Notice that the relation $(w, h) \in [T_{f(0)}]$ admits a scale (see Section 2.3). Also,

$$v \in [T_{f(0)} \restriction \{f(0), f(1), \dots\}] \iff \exists x \in \omega^\omega \quad (v, (f \circ x)) \in [T_{f(0)}].$$

Then using the fact that for a tree S on ω, there exists a tree $\hat{S}$ on $\omega \times \omega$ such that $v \notin [S] \iff v \in p[\hat{S}]$, we have that $v \notin [T_{f(0)} \restriction \{f(0), f(1), \dots\}]$ admits a scale and so $\forall v \left[v \notin [T_{f(0)} \restriction \{f(0), f(1), \dots\}]\right]$ admits a scale by the 2nd periodicity theorem of Moschovakis [M2]. By the definition of λ being reliable, we have that $\varphi_0(v) \leq \varphi_0(w)$ admits a scale. Finally, using the fact that if A and B admit scales then $A \cap B$ and $A \cup B$ also admit scales, we have that R admits a scale. A similar argument shows that $\neg R$ admits a scale and so this game is determined.

Our second claim is that player II wins this game. Suppose, towards a contradiction, that player I had a winning strategy and played $f(0)$ as her first move. Player II may then fix a $w \in W$ so that $\varphi_0(w) = f(0)$. Then player II would simply enumerate w while also playing $h(0) = \varphi_0(w), h(1) = \varphi_1(w), \dots$ etc. The second condition of a win for player II is guaranteed since

$$\forall v \in p[T_{f(0)}] \ \ \varphi_0(v) \leq f(0). \tag{6.1}$$

To see that 6.1 is true, consider $v \in p[T_{f(0)}]$. Then there is some $\vec{\alpha} \in \lambda^\omega$ so that

$$\forall n \in \omega \ \ ((v(0), \dots, v(n)), (\alpha(0), \dots, \alpha(n))) \in T \ \text{ and } \ \alpha(0) = f(0).$$

So

$$\begin{array}{lll} & \text{for } n = 0 & \exists v_0 \supset (v(0)), \varphi_0(v_0) = f(0) \\ \text{and} & \text{for } n = 1 & \exists v_1 \supset (v(0), v(1)), \varphi_0(v_1) = f(0) \\ \text{and} & \text{for } n = 2 & \exists v_2 \supset (v(0), v(1), v(2)), \varphi_0(v_2) = f(0) \\ \vdots & \vdots & \end{array}$$

By the lower semi-continuity property of scales, $\varphi_0(v) \le f(0)$. Thus, player II beat player I's strategy.

Let $G_0 : \lambda^\omega \to \omega^\omega$ be the function produced by player II's strategy where the ordinal moves are ignored. Then if $\{f(0), f(1), \dots\}$ is $f(0)$-honest, there is some $w \in W$ so that $\varphi_0(w) = f(0)$ and for each $i \in \omega$, $\varphi_i(w) \in \{f(0), f(1), \dots\}$. Letting $v = G_0(f)$, we have, by our second condition of player II's strategy, and 6.1 above, that

$$\varphi_0(w) \le \varphi_0(v) \le f(0)$$

and so $\varphi_0(v) = f(0)$. In other words, $\varphi_0(G_0(f)) = f(0)$.

To show the second part of the theorem, we will again play a game on ordinals and integers,

$$\begin{array}{c|cccccc} \mathrm{I} & f(0) & & f(1) & & f(2) & \cdots \\ \mathrm{II} & & w_0, h(0) & & w_1, h(1) & & \cdots \end{array}$$

In this case, however, we define the payoff of the game by saying that player II wins $\iff \forall n \in \omega$

1. $(w, h)_n \in [T_{(f)_n(0)}]$ and
2. $\forall v \left[(v)_n \in [T_{(f)_n(0)} \restriction \{f(0), f(1), \dots\}] \Rightarrow \varphi_0((v)_n) \le \varphi_0((w)_n)\right]$.

Using similar arguments we can show that this game is determined, that player II does win this game and that the desired function results. ∎

Definition 6.7 *A cardinal κ is said to be Suslin if there exists a scale $\{\varphi_i\}_{i\in\omega}$, and a set $W \subset \omega^\omega$, such that $\{\varphi_i\}_{i\in\omega}$ is a scale into κ and for all $\lambda < \kappa$, $\{\varphi_i\}_{i\in\omega}$ is not a* *scale on W into λ.*

The following theorem is due to Jackson[J2].

Theorem 6.8 ($\mathbf{AD} + \mathbf{V} = \mathbf{L}[\mathbb{R}]$) *Let κ be a Suslin cardinal and μ_κ the supercompactness measure on $P_{\omega_1}(\kappa)$. Then*

$$j_{\mu_\kappa}(\omega_1) = \kappa^+.$$

Proof. By Lemma 4.3 we have that $j_{\mu_\kappa}(\omega_1) \geq \kappa^+$, thus we need only show that $j_{\mu_\kappa}(\omega_1) \leq \kappa^+$.

Let $\{\varphi_i\}_{i\in\omega}$ be a scale on $W \subset \omega^\omega$ into κ and let $F : P_{\omega_1}(\kappa) \to \omega_1$ and play the following game G.

$$\begin{array}{c|ccccccc} \mathrm{I} & f(0) & & f(2) & & f(4) & \cdots \\ \mathrm{II} & & f(1), x_0 & & f(3), x_1 & & \cdots \end{array}$$

where $f(i) \in \kappa$ and $x_i \in \omega$. Player II wins $\iff$ $x = (x_0, x_1, \dots)$ codes a well-ordering of ω and $|x| \geq F(\hat{S})$ where $\hat{S} = \{\varphi_0((G(f))_0), \varphi_0((G(f))_1), \dots\}$.

This game is determined (see [KW]) and player II has a winning strategy. To see this, suppose that player I won following a strategy, σ. Let S be an honest set in $P_{\omega_1}(\kappa)$ closed under σ. Then player II need merely enumerate S and play a real which codes a well-ordering of ω greater than $F(S)$ and player II will have won, a contradiction.

Let $\mathcal{F} : \kappa^\omega \to \omega^\omega$, be the function coming from player II's winning strategy where player II's ordinal moves are ignored. Then $\forall S \in P_{\omega_1}(\kappa)$, honest and closed under $\mathcal{F}$, and $\forall s$ enumerating S, $\mathcal{F}(s)$ is a well-ordering of ω of length $\geq F(S)$. Notice that it is the case that $\forall^*_{\mu_\kappa} S \quad S$ is honest and closed under $\mathcal{F}$. Also, for a fixed S, which we may view as ω, we abbreviate the statement, "For comeager many $s \in S^\omega$ extending the finite sequence, p," by "$\forall^*_p s \in S^\omega$". For $a, b \in \omega$ and $s \in \kappa^\omega$ we let, "$a <_{\mathcal{F}(s)} b$," abbreviate the statement, "a is less than b with respect to the well-ordering $\mathcal{F}(s)$ on ω". Similarly, $|a|_{\mathcal{F}(s)}$ is defined to be the length of a with respect to $\mathcal{F}(s)$.

We define a tree T on $\kappa^{<\omega} \times \omega$ as follows:

$$((p_0, p_1, \ldots, p_n)(a_0, a_1, \ldots, a_n)) \in T \iff$$

1. $p_i \in \kappa^{<\omega}, a_i \in \omega$ and $\forall 0 \leq i \leq n-1 \quad p_{i+1} \supset p_i$
2. $\forall 0 \leq i \leq n-1 \quad \forall^*_{\mu_\kappa} S \quad \forall^*_{p_{i+1}} s \in S^\omega \quad a_{i+1} <_{\mathcal{F}(s)} a_i$
3. $\forall 0 \leq i \leq n \quad \exists \beta_i : P_{\omega_1}(\kappa) \to \omega_1 \quad \forall^*_{\mu_\kappa} S \quad \forall^*_{p_i} s \in S^\omega \quad |a_i|_{\mathcal{F}(s)} = \beta_i(S)$

Notice that T is well-founded, otherwise we could use the definition of T, and the countable additivity of μ_κ to produce an infinitely decreasing sequence of ordinals. Furthermore, since T is a tree on $\kappa^{<\omega} \times \omega$, $|T| < \kappa^+$.

It must now be shown that $[F] \leq |T|$. Suppose, to the contrary, that $[F] > |T|$. Then we may fix $H_0 : P_{\omega_1}(\kappa) \to \omega_1$ so that $[H_0] = |T|$. Now $\forall^*_{\mu_\kappa} S \quad F(S) > H_0(S)$. Fix such an S. Then for any enumeration s of S, we have that $\mathcal{F}(s)$ is a well-ordering of ω of length $> H_0(S)$. For each such s there is some integer a_s so that $|a_s|_{\mathcal{F}(s)} = H_0(S)$. Since there are non-meager many enumerations and a countable union of meager sets is meager, we have that there is some $a_S \in \omega$ so that for non-meager many $s \quad |a_S|_{\mathcal{F}(s)} = H_0(S)$ and so there is a neighborhood on which the set of s for which $|a_S|_{\mathcal{F}(s)} = H_0(S)$ is comeager, and this neighborhood is defined by some

$p \in S^\omega$. Using the normality of μ_κ we have that there is some $p_0 \in \kappa^{<\omega}$ so that $\forall^*_{\mu_\kappa} S \;\; \forall^*_{p_0} s \in S^\omega \;\; |a_S|_{\mathcal{F}(s)} = H_0(S)$. The countable additivity of μ_κ may be exploited to obtain some $a_0 \in \omega$ so that $\forall^*_{\mu_\kappa} S \;\; \forall^*_{p_0} s \in S^\omega \;\; |a_0|_{\mathcal{F}(s)} = H_0(S)$.

Now $|((p_0)(a_0))|_T < |T|$ so there is some $H_1 : P_{\omega_1}(\kappa) \to \omega_1$ such that $[H_1] = |((p_0)(a_0))|_T$. An argument similar to that just given yields a pair (p_1, a_1) so that $((p_0, p_1)(a_0, a_1)) \in T$ and

$$\forall^*_{\mu_\kappa} S \;\; \forall^*_{p_1} s \in S^\omega \;\; |a_1|_{\mathcal{F}(s)} = H_1(S).$$

Obviously, this argument will continue indefinitely and along the way, condition 3 guarantees that $\forall i \in \omega$ we may define $\beta_i : P_{\omega_1}(\kappa) \to \omega_1$ so that

$$[\beta_0] > [\beta_1] > [\beta_2] > \cdots$$

which is, of course, a contradiction. Thus $[F] \leq |T| < \kappa^+$ and so $j_{\mu_\kappa}(\omega_1) \leq \kappa^+$. ∎

6.3 Conditions Related to the Upper Bound

One of the problems associated to finding an upper bound for $j_{\mu_m}(\omega_n)$ $(m > 2)$ is that unlike μ_1 and μ_2, there is no known characterization of μ_m based on ordinal measures. If such a characterization existed then partition arguments could most likely be applied and the upper bound attained. The following discussion will provide a partial basis for this view.

Definition 6.9 *A function $F : P_{\omega_1}(\omega_m) \to \omega_1$ is order type bounded (o.t. bdd) if and only if there exists a function $f : \omega_1 \to \omega_1$ such that*

$$\forall^*_{\mu_m} S \;\; F(S) \leq f(o.t.(S)).$$

Proposition 6.10 *The following are equivalent:*

1. $j_{\mu_m}(\omega_1) \leq \omega_{m+1}$

2. $\forall F : P_{\omega_1}(\omega_m) \to \omega_1 \;\; F$ *is o.t. bdd*

3. $\forall F : P_{\omega_1}(\omega_m) \to \omega_1$ *there is a well-ordering* $\ll$ *on* ω_1^m *such that*

$$\forall^*_{\mu_m} S \;\; F(S) \leq |\ll\restriction S|$$

Proof.

$(1 \Rightarrow 2)$

Fix $F : P_{\omega_1}(\omega_m) \to \omega_1$ and let $g : \omega_1^m \to \omega_1$ be such that $[F]_{\mu_m} < [g]_{W_1^m}$. Such a g is guaranteed by the hypothesis. Define the function, $G : P_{\omega_1}(\omega_m) \to \omega_1$ by

$$G(S) = g(\alpha_S^1, \alpha_S^2, \ldots, \alpha_S^m)$$

where each α_S^i is defined as in Theorem 4.2. Then $[G]_{\mu_m} \geq [g]_{W_1^m}$. Thus $[G]_{\mu_m} > [F]_{\mu_m}$ and so

$$\forall^*_{\mu_2} S \;\; F(S) < G(S) = g(\alpha_S^1, \ldots, \alpha_S^m) \leq g(1)(\alpha_S^m).$$

Thus $f : \omega_1 \to \omega_1$ defined by $f(\alpha) = g(1)(\alpha)$ is the desired bounding function.

$(2 \Rightarrow 3)$

Fix $F : P_{\omega_1}(\omega_m) \to \omega_1$ and let f be a bounding function. By Kunen's Theorem, since $f : \omega_1 \to \omega_1$, there is some well-ordering $\prec$ of ω_1 so that $\forall^*_{W_1^1} \alpha \;\; f(\alpha) < |\prec\restriction \alpha|$. We define a well-ordering $\ll$ on ω_m by

$$\eta_1 \ll \eta_2 \iff \forall^*_{\mu_m} S \;\; C_S(\eta_1) \prec C_S(\eta_2)$$

where $C_S(\eta)$ is the unique element $\beta < \omega_1$ to which η collapses as S is collapsed. Easily, $\ll$ is a well-ordering on ω_m. Let

$$\mathcal{A} = \{S \in P_{\omega_1}(\omega_m) \mid \forall \eta_1, \eta_2 \in S \ \ (C_S(\eta_1) \prec C_S(\eta_2) \iff \eta_1 \ll \eta_2)\}.$$

Suppose that $\mathcal{A} \notin \mu_m$. Then

$$\forall^*_{\mu_m} S \ \ \exists \eta_1, \eta_2 \in S \ (C_S(\eta_1) \prec C_S(\eta_2) \text{ and } \eta_2 \ll \eta_1).$$

For each such S we may fix a least such pair (η_1^S, η_2^S) and then by normality get $(\hat{\eta}_1, \hat{\eta}_2)$ so that

$$\forall^*_{\mu_m} S \ \ (C_S(\hat{\eta}_1) \prec C_S(\hat{\eta}_2) \text{ and } \hat{\eta}_2 \ll \hat{\eta}_1)$$

but this is absurd. So $\mathcal{A} \in \mu_m$.

Fix $S \in \mathcal{A}$. For $\eta_1, \eta_2 \in S$,

$$\eta_1 \ll \eta_2 \iff C_S(\eta_1) \prec C_S(\eta_2)$$

and thus $| \prec \restriction o.t.(S)| = | \ll \restriction S|$. So

$$\forall^*_{\mu_m} S \ \ F(S) < | \ll \restriction S|.$$

$(3 \Rightarrow 1)$

Fix $F : P_{\omega_1}(\omega_m) \to \omega_1$ and let $\ll$ be a well-ordering of ω_m satisfying the hypothesis. Fix $G : P_{\omega_1}(\omega_m) \to \omega_1$ such that $[G] < [F]$. Then $\forall^*_{\mu_m} S \ \ G(S) < F(S) < | \ll \restriction S|$. Fix an S so that the above inequality is satisfied. Then there is an ordinal $\gamma_S^G \in S$ so that $G(S) = |\gamma_S^G|_{\ll \restriction S}$ where $|\gamma_S^G|_{\ll \restriction S} = | \ll \restriction \{\eta \in S \mid \eta \ll \gamma_S^G\}|$. By normality, there is an ordinal γ^G so that

$$\forall^*_{\mu_m} S \ \ G(S) = |\gamma^G|_{\ll \restriction S}.$$

The mapping $G \mapsto \gamma^G$ is easily well defined and order preserving and thus an embedding from $[F]$ to $(\omega_m, \ll)$. Thus $| \ll | \geq [F]$. Since $| \ll | < \omega_{m+1}$ we have that $[F] < \omega_{m+1}$ and so $j_{\mu_m}(\omega_1) \leq \omega_{m+1}$. ∎

Lemma 6.11 *Fix $n \in \omega$ and suppose that*

$$\forall m \leq n \ \ \forall F : P_{\omega_1}(\omega_m) \to \omega_1 \ \ F \text{ is o.t. bdd}$$

then $\forall F : P_{\omega_1}(\omega_n) \to \omega_1 \ \ \exists f : \omega_n \to \omega_1$ such that $\forall^*_{\mu_n} S \ \ F(S) = f(\alpha^1_S, \ldots, \alpha^n_S)$ where, as usual, $\alpha^i_S = o.t.(S \cap \omega_i)$.

Proof. Fix $F : P_{\omega_1}(\omega_n) \to \omega_1$ and let $f_n : \omega_1 \to \omega_1$ be a bounding function. By Kunen's Theorem there is a well-ordering $\prec_n$ of ω_1 such that $\forall^*_{W^1_1} \alpha \ \ f_n(\alpha) < | \prec_n \restriction \alpha|$. Then

$$\forall^*_{\mu_n} S \ \ F(S) < | \prec_n \restriction \alpha^n_S|.$$

For each such S let γ_S be the element of S so that $F(S) = |C_S(\gamma_S)|_{\prec_n \restriction \alpha^n_S}$ where $C_S(\gamma_S)$ is the ordinal in ω_1 to which γ_S collapses. By normality there is some γ_n so that

$$\forall^*_{\mu_n} S \ \ F(S) = |C_S(\gamma_n)|_{\prec_n \restriction \alpha^n_S}.$$

Consider the function $S \mapsto C_S(\gamma_n)$. This is a function taking $P_{\omega_1}(\omega_n)$ to ω_1 such that

$$\forall S \in P_{\omega_1}(\omega_n) \ \ (\gamma_n \in S \Rightarrow C_S(\gamma_n) = o.t(S \cap \gamma_n)).$$

Fixing a bijection $\rho : \omega_{n-1} \to \gamma_n$ we define the function $H_{n-1} : P_{\omega_1}(\omega_{n-1}) \to \omega_1$ by $H_{n-1}(S) = o.t(\rho''(S))$. This function is o.t. bdd so there is some bounding function f_{n-1} and consequently some well-ordering $\prec_{n-1}$ so that

$$\forall^*_{\mu_{n-1}} S \ \ H_{n-1}(S) < | \prec_{n-1} \restriction \alpha^{n-1}_S|.$$

As before we may obtain an ordinal γ_{n-1} so that

$$\begin{aligned} &\forall^*_{\mu_{n-1}} S \quad H_{n-1}(S) = |C_S(\gamma_{n-1})|_{\prec_{n-1}\restriction\alpha_S^{n-1}} \\ \Rightarrow \quad &\forall^*_{\mu_n} S \quad F(S) = ||C_S(\gamma_{n-1})|_{\prec_{n-1}\restriction\alpha_S^{n-1}}|_{\prec_n\restriction\alpha_S^n} \end{aligned}$$

We may continue this process until we obtain a function $H_1 : P_{\omega_1}(\omega_1) \to \omega_1$ so that

$$\forall^*_{\mu_1} S \quad H_1(S) = |C_S(\gamma_1)|_{\prec_1\restriction\alpha_S^1}.$$

Recall that μ_1 is just W_1^1 and in this case $\forall^*_{\mu_1} S \quad C_S(\gamma_1) = \gamma_1$ so that

$$\forall^*_{\mu_n} S \quad F(S) = |\cdots||\gamma_1|_{\prec_1\restriction\alpha_S^1}|_{\prec_2\restriction\alpha_S^2}\cdots|_{\prec_n\restriction\alpha_S^n}.$$

We associate to the function F a function $f : \omega_1^n \to \omega_1$ defined by

$$f(\alpha_1, \alpha_2, \ldots, \alpha_n) = |\cdots||\gamma_1|_{\prec_1\restriction\alpha_1}|_{\prec_2\restriction\alpha_2}\cdots|_{\prec_n\restriction\alpha_n}.$$

Then $\forall^*_{\mu_n} S \quad F(S) = f(\alpha_S^1, \ldots, \alpha_S^n)$. ∎

Corollary 6.12 *If $j_{\mu_m}(\omega_1) \leq \omega_{m+1}$ for all m then*

$$\forall F : P_{\omega_1}(\omega_m) \to \omega_1 \quad \exists f : \omega_m \to \omega_1 \quad \forall^*_{\mu_m} S \quad F(S) = f(\alpha_S^1, \ldots, \alpha_S^m).$$

Proof. By Proposition 6.10 we have that $\forall m \quad \forall F : P_{\omega_1}(\omega_m) \to \omega_1 \quad F$ is o.t. bdd. The desired result then follows from the previous lemma. ∎

What has been shown is that if an upper bound exists for $j_{\mu_m}(\omega_1)$, $m \in \omega$, then we have a complete characterization of the functions taking $P_{\omega_1}(\omega_m)$ to ω_1. Since the upper bound does exist for $j_{\mu_2}(\omega_1)$ we know that any function $F : P_{\omega_1}(\omega_2) \to \omega_1$ has such a characterization.

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